A connection between probability and fuzzy logic by means of dialogue games

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Structure of the talk:

1. T-norm based fuzzy logics
2. Dutch book argument for probability
3. Dialogue games
4. Giles games for Ł
5. Games for Π, G (Fermüller)
6. Probabilistic frames for all t-norm fuzzy logics
7. Conclusions

Fuzzy logic

Strategy:

- Generalize bivalent classical logic to $[0, 1]$
- Impose some restrictions on the truth functions of the connectives
- Derive the semantics and axioms from these postulates

Design choices:

- Truth-functionality of all connectives w.r.t. $[0, 1]$
- Start with conjunction & and require some natural conditions
  ... continuous t-norms
- Other connectives determined by & in a natural way

The conditions on &

Commutativity: $x * y = y * x$

- When asserting two propositions, it does not matter in which order we put them down
- The commutativity of classical conjunction seems not affected by taking into account also fuzzy propositions

Associativity: $(x * y) * z = x * (y * z)$

- When asserting three propositions, it is irrelevant which two of them we put down first (be they fuzzy or not)

Monotony: $x \leq x' \;\Rightarrow\; x * y \leq x' * y$

- Increasing the truth value of the conjuncts should not decrease the truth value of their conjunction

Classicality: $x * 1 = x,\ x * 0 = 0$

- $0, 1$ represent the classical truth values for crisp propositions
- Fuzzy logic generalizes, not replaces classical logic

Continuity: $*$ *continuous*

- An infinitesimal change of the truth value of a conjunct should not radically change the truth value of the conjunction

We could add further conditions on & (e.g., idempotency), but it has proved convenient to stop here, as it already yields a rich and interesting theory

$\Rightarrow$ The truth function of & . . . a continuous t-norm

Three important t-norms:

- Gödel t-norm G . . . $x * y = \min(x, y)$
- Łukasiewicz t-norm Ł . . . $x * y = \max(0, x + y - 1)$
- Product t-norm Π . . . $x * y = x \cdot y$

Truth functions of other connectives:

- $(x \rightarrow y) =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$ (the adjoint functor to &)
  (maximal function for internalized modus ponens)
- $\neg x =_{\mathrm{df}} x \rightarrow 0$ (reductio ad absurdum)
- $(x \leftrightarrow y) =_{\mathrm{df}} (x \rightarrow y) \,\&\, (y \rightarrow x)$ (bi-implication)
- min, max (turn out to be definable)
  $(x \wedge y) =_{\mathrm{df}} x * (x \rightarrow y)$
  $(x \vee y) =_{\mathrm{df}} ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$

Semantics of main t-norm connectives:

	G	Ł	Π
$x \mathbin{\&} y$	$\min(x, y)$	$\max(0, x + y - 1)$	$x \cdot y$
$x \to y$ if $x \leq y$	1	1	1
$x \to y$ if $x > y$	y	$\min(1, 1 - x + y)$	y/x
$\neg x$	$1 - \mathrm{sgn}(x)$	$1 - x$	$1 - \mathrm{sgn}(x)$

$(x \leftrightarrow y) = 1$ iff $x = y$

Evaluation = assignment of particular values in $[0, 1]$ to propositional variables

Evaluation of formulae ... compositionally
= a straightforward generalization of Tarski's conditions to $[0, 1]$

$*$-tautology ... $e(\varphi) = 1$ for every evaluation e of atoms

Logic PC($*$) ... the set of all $*$-tautologies

The logics of the three important t-norms:

- Gödel logic G
- Łukasiewicz logic Ł
- Product logic Π

Some formulae are $*$-tautologies for any continuous t-norm $*$
... *t-tautologies* (e.g., $\varphi \to \varphi$)

Hájek's Basic (Fuzzy) Logic BL ... the set of all t-tautologies

BL $\subset$ PC($*$) $\subset$ Bool

The logics of continuous t-norms proved to be axiomatizable:

BL:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$(\varphi \,\&\, \psi) \to \varphi$$
$$(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$$
$$((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$$
$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
$$0 \to \varphi$$

G: $\varphi \to (\varphi \,\&\, \varphi)$

Ł: $\neg\neg\varphi \to \varphi$

Π: $\neg\neg\varphi \to ((\varphi \to (\varphi \,\&\, \psi)) \to (\psi \,\&\, \neg\neg\psi))$

PC($*$): ...

+ deductive rule *modus ponens:* from φ, $\varphi \to \psi$ infer ψ

Characterization of continuous t-norms

Theorem (Mostert–Shields, 1957): Each continuous t-norm is an ordinal sum of isomorphic copies of G, Ł, and Π.

- Idempotent elements form a closed set . . . G
- The intervals between are isomorphic to Ł or Π
- Values from different intervals evaluate as in G

Notation: $Ł \oplus \Pi$, $Ł \oplus \Pi \oplus \Pi$, . . .

Theorem: $BL = Ł \oplus Ł \oplus Ł \oplus \ldots$

Simple Giles game – atomic propositions

- two players (Me and You, Proponent and Opponent) are betting on results of some *yes/no experiment* (‘ the spin of the particle will be +’)
- each event E (result of experiment) is expressed by an atomic proposition e and has certain (objective) probability of occurrence, or dually a *risk value* <e>*
- a bet is given by a multiset of your events (propositions) $f_1, ..., f_n$ against a multiset of my events $e_1 .... e_m$

$$[f_1 .... f_n \mid e_1 .... e_m]$$

Simple Giles game – payoffs

- payoffs for the game $[f_1, ..., f_n \mid e_1, ..., e_m]$
- I pay you 1€ for each of my events $e_1, ..., e_m$ which does not occur
- you pay me 1€ for each of your events $f_1, ..., f_n$ which does not occur
- the payoff for the empty multiset of events [|...] or [...|] is 0
- the *game is fair* (from my point of view) if my total risk is not greater than yours

$$\Sigma_I \langle f_i \rangle^* \geq \Sigma_j \langle e_j \rangle^*$$

Giles game – complex propositions

- in general there is no straightforward correspondence between (compound) propositions of a fuzzy logic and events
- payoffs are defined just on the basis of events
- bets on complex propositions shall be *transformed* onto bets on atomic propositions
- we shall have a *fair transformation rule* which transforms fair bets onto fair bets
- instead of the correspondence between *propositions* (*formulae*) *of classical logic* and *events* we have a correspondence between *propositions of fuzzy logic* and *bets on events*

Giles game for Ł – the transformation rule

- the connectives in Ł are interdefinable, we define the game for the material implication → and the constant for contradiction ⊥
- ⊥ corresponds to the atomic event which never happens (impossible event)

The '→' rule:

- You can *attack* my bet on $A \rightarrow B$ by betting on A and forcing me to bet on B
- You may explicitly refuse to attack $A \rightarrow B$

$$[e_1, \ldots, e_{m-1} \mid A \rightarrow B, f_1, \ldots, f_n]$$

$$[e_1, \ldots, e_{m-1}, \mid f_1, \ldots, f_n] \ (\textit{refuse}) \qquad [e_1, \ldots, e_{m-1}, A \mid f_1, \ldots, f_n, B] \ (\textit{attack})$$

- I can attack your bet the same way
- '→' rule is the same as in a dialogue game for classical logic (Lorenzen 1950's)
- implication is a conditional betting rule: 'if you bet on A, I am ready to bet on B'

Giles game for Ł

- the game starts in a state [| *F*] i.e. with Myself betting on some proposition *F*
- the game ends if there are no compound propositions
- the game is *a win for me* if I have a strategy to end the game in a position where my risk is not greater than your risk

Correspondence theorem (Giles 1958, Fermüller 2005)

I have a winning strategy for the Giles game [| *F*] for a given assignment of risk values <> iff *F* is a true statement of Lukasiewicz logic under the assignment of (fuzzy) values corresponding to <>. Analogously, I have a winning strategy for the game [| *F*] for any assignment of risk values iff *F* is a tautology of Lukasiewicz logic.

Giles game for Π and G - rules

- primitive connectives: &, $\rightarrow$, $\bot$
- initial state: [|*F*], terminal states: all e_i's and f_j's atomic
- arbitrary order of turns

Opponent's turns:

[*F* |*G*,*A* & *B*] just rewrite as [*F* |*G*,*A*, *B*]

[*F* |*G*,*A* $\rightarrow$ *B*]

- Opponent grants *A* $\rightarrow$ *B*: continue with [*F* |*G*]
- Opponent attacks *A* $\rightarrow$ *B*:
  - (a+) Proponent concedes: continue with [*F*,*A* |*G*,*B*]
  - (a-) Proponent insists on validity: continue with [*A* | *B*]

Proponent's turns:

dual (dtto on the right-hand side) Notice the *role switch* in (a-) here: implication is defended by the player who insists on it

Giles game for Π and G - payoffs

- the rules of the game are uniform for all three logics, for L the extension of the implication clause is trivial
- they differ in calculating payoffs: in the terminal state $[f_1, \ldots, f_n \,|\, e_1, \ldots, e_m]$

The payoffs for G: $\min\langle f_i \rangle \leq \min \langle e_j \rangle$

"My least probable event is more probable than that of yours."

The payoffs for Π: $\Pi\langle f_i \rangle \leq \Pi\langle e_j \rangle$

"Probability of all of my events happening is greater than that for you."

The payoffs for L: $1-\sum(1-\langle f_i \rangle) \leq 1-\sum(1-\langle e_j \rangle)$

ie. $\sum(1-\langle f_i \rangle) \geq \sum(1-\langle e_j \rangle)$

Probabilistic justification of fuzzy logics

Evaluation of terminal sequents:

Łukasiewicz:	we *sum* the inverted results	(cf. Ł)
Gödel:	we take the *minimum*	(cf. G)
Product:	we take the *product*	(cf. Π)

Questions:

- Can we generalize the results to other PC($*$)'s and BL?
- Can the formula be interpreted as a bet on some event?

Problem:

Fuzzy logic is truth-functional w.r.t. $[0,1]$
Probability is not truth-functional w.r.t. $[0,1]$
(events need not be independent)

However, there are systems of events on which probability is truth-functional w.r.t. $[0,1]$ (for non-identical events); we shall call them truth-functional frames

Examples:

- A_i independent $\Rightarrow \mathsf{P}(A_i \cap A_j) = \mathsf{P}(A_i) \cdot \mathsf{P}(A_j)$ (cf. Π)
- A_i disjoint $\Rightarrow \mathsf{P}(A_i \cup A_j) = \mathsf{P}(A_i) + \mathsf{P}(A_j)$ (cf. Ł)
- A_i a $\subseteq$-chain $\Rightarrow \mathsf{P}(A_i \cap A_j) = \min(\mathsf{P}(A_i), \mathsf{P}(A_j))$ (cf. G)

First approximation: To get a fuzzy logic, evaluate probabilistically in a truth-functional frame

Why the first approximation does not work:

1. The evaluation is still not fully truth-functional:
$$\mathsf{P}(A_i \cap A_i) = \mathsf{P}(A_i)$$

2. Only (strong) conjunction and disjunction, involutive negation, 0 and 1 can be expressed by means of set-operations on the frame; fuzzy implication does not correspond to any set-operation $\Rightarrow$ we get only a *fragment* of fuzzy logic

3. Truth-functional frames only give a sound, *not complete* semantics of fuzzy logics. Example: $\mathsf{P}(A_i \cap A_j) = 0$ is true for all disjoint events, but $(\varphi \,\&\, \psi) \leftrightarrow 0$ is not a theorem of Ł

The failures are remedied by the dialogue–betting game:

The complex formula has been decomposed by the dialogue game according to the meanings of connectives to an 'equivalent' bet $[F \mid G]$ involving only atoms

The bet $[F \mid G]$ is interpreted as follows:

- Atoms are event types (multiple occurrences of the same variable are assigned different events, but with the same probability)
- Proponent bets on the event $\bigcap G$, Opponent on $\bigcap F$
- Proponent wins iff $P(\bigcap G) \geq P(\bigcap F)$

The choice of the truth-functional frame determines the resulting fuzzy logic:

Π: independent events
Ł: complements of disjoint events
G: comparable events

Theorem: There is a frame for any logic of a continuous t-norm

Hint: The frame is a combination of the above three by the Mostert-Shields decomposition of the t-norm.

$\Rightarrow$ Fuzzy logic is a way of reasoning about probabilities of limited sets of events (independent, disjoint, comparable, . . .)

Dutch book and Giles game

To compare fuzzy and probabilistic games we use the framework of the game theory. A Dutch Book can be seen as a nondeterministic two-person game:

- Bettor – assigns probabilities (quotients) $q_1,\ldots, q_n$ to a finite set of events $E_1, \ldots E_n$
- Bookie – sets the signs $sgn_1, \ldots sgn_n$ for $q_1,\ldots, q_n$
- Nature's move – makes (some of) the events $E_1, \ldots E_n$ happen
- the payoff $\Sigma \mathrm{sgn}_i(||E_i||*S-q_i*S)$

The bet $q_1,\ldots, q_n$ on $E_1, \ldots E_n$ is said to be *Dutch bookable* iff the Bookie has a strategy (i.e. a choice of signs $sgn_1, \ldots sgn_n$) such that for any move of the Nature (any subset of $E_1, \ldots E_n$ happening) his payoff is positive, i.e. a strategy which leads to an *immediate loss* of the Bettor (The DB-game is non zero sum.)

Dutch book and Giles game

Giles game for tautologies

- Opponent chooses the assignment *v* of atoms
- the game for *F* with the assignment *v* is played
- payoffs

The Giles bet on *F, v* is said to be *unfair* if the Opponent has a strategy such that the Proponent's *expected* loss is greater than the Opponents one.

Dutch book and Giles game

The penalty for inconsistency in a DB is stronger than the one in GB (sure loss x expected loss). The only immediate loss in GB is when betting on contradiction.

- in DB game this means loss in any play no matter what is Nature's move
- in GG the situation is different – the inconsistence with respect to tautologicity leads to an expected loss