Fully Probabilistic vs Bayesian DM

Miroslav Kárný

Adaptive Systems Department
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
school@utia.cas.cz

Theme of talk

Domain Decision Making (DM) under uncertainty, incomplete knowledge and limited ability to evaluate

B DM Bayesian DM theory defining the optimal DM strategy

${}^{O}R$ = Arg min $E[Z]$ = minimum of expected loss

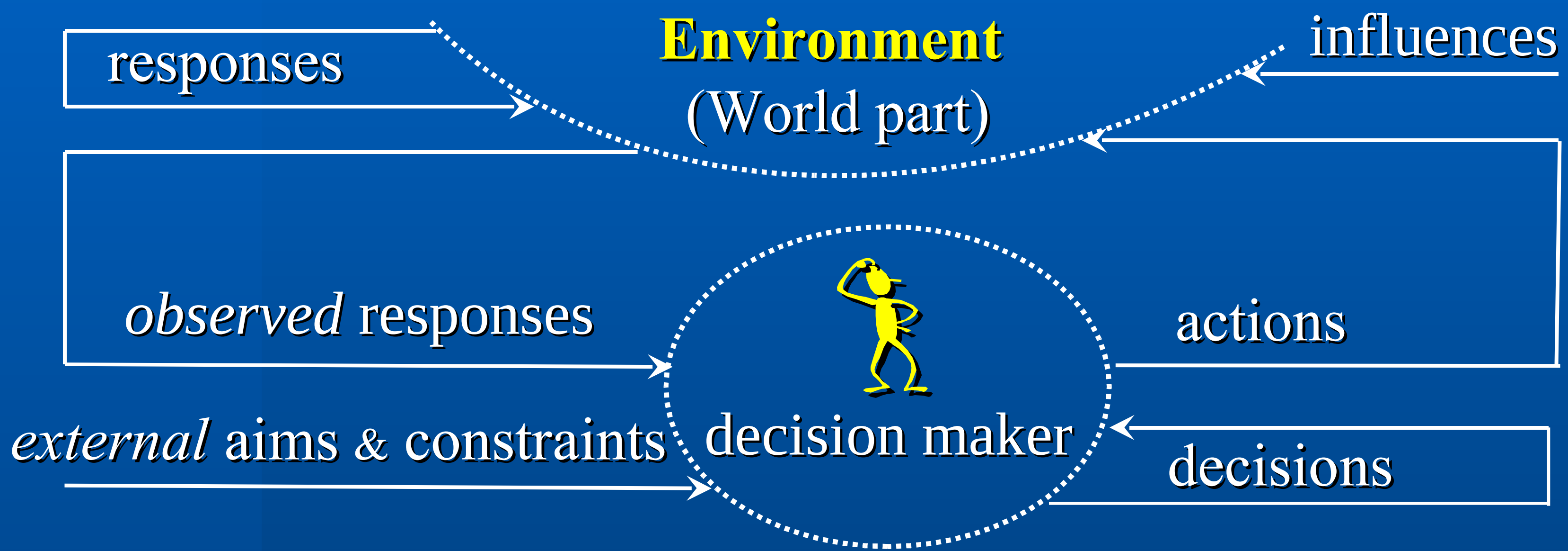
while describing behavior Q of DM loop by probability density function (pdf) $f(Q)$

FP DM Fully probabilistic DM theory defining the optimal strategy

R = Arg min $D[f\,||\,{}^{I}f\,]$ = KLD of $f\,(Q)$ on ideal pdf ${}^{I}f\,(Q)$ expressing DM aims and constraints

Relationship of B DM of FP DM ?

DM structure and elements



- provides *internal* aims & constraints
- designs & applies DM strategy *R*: data → actions

Data knowledge, observations, goal & constraint descriptions

Behavior *Q* *considered* actions, observations & internals, i.e., unobserved influences and responses

Bayesian and Fully Probabilistic DM

Optimal B DM strategy ${}^{O}R$: $\quad \text{Arg min} \underbrace{\int Z(Q)\, f(Q) dQ}_{\text{expected loss } E[Z]}$

Closed-loop model $f(Q)$, conditioned on prior knowledge, factorizes

$$
\begin{aligned}
f(Q) &= f\,(\text{observations}, \textit{actions}, \text{internals}) \\
&= \underbrace{f\,(\text{observations, internals} \mid \textit{action}, \text{data})}_{\text{chosen environment model}} \times \underbrace{f\,(\textit{action} \mid \text{data})}_{\text{optimized strategy}}
\end{aligned}
$$

FP DM: $\quad Z(Q) = \ln\big(\underbrace{f(Q) \,/\, {}^{I}f(Q)}_{\text{aims- constraints- expressing ideal pdf}} \big) \;\Leftrightarrow\; E[Z] = \underbrace{\int f(Q) \ln\big(f(Q) \,/\, {}^{I}f(Q)\big)\, dQ}_{\text{KL divergence}}$

Relationships of B DM of FP DM ?

Basis of DM under uncertainty revisited

Quest for optimality

- strict partial ordering $<_{Q^*}$ of behaviors $Q \in Q^*$ is assumed to exist
- complete ordering $<_{R^*}$ of strategies $R \in R^*$ is searched for that
  - is based on well-specified assumptions
  - respects ordering $<_{Q^*}$ of behaviors $Q \in Q^*$
  - fully exploits knowledge available
  - serves to all DM tasks with common information structure
  - is generated by a technique avoiding unjustified restrictions

Towards strategy ordering (fixed environment)

Prop $\boxed{\exists \text{ non-unique loss } Z: Q^* \to [-\infty,\infty]: Q_1 <_{Q^*} Q_2 \Rightarrow Z(Q_1) < Z(Q_2)}$

{under general topological conditions, Fishburn 1970}

Def Behavior Q decomposes symbolically to (Q_R, N):

Q_R a known constituent or determined by the used strategy R

$N \in N^* \neq \emptyset$ uncertainty, i.e., unknown constituent independent of R

Def Functions $Z_R(N) \in Z_{R^*} = \{Z_R(N) = Z(Q_R, N), R \in R^*\}$ of $N \in N^* \neq \emptyset$, gained from the loss $Z(Q)$ for various strategies $R \in R^*$, are ordered partially by the dominance ordering

$$Z_{R1} < Z_{R2} \Leftrightarrow Z_{R1}(N) \leq Z_{R2}(N),\ \forall N \in N^*, \text{ sharp for enough } N$$

Prop $\exists$ non-unique "loss" $T: Z_{R^*} \to [-\infty,\infty]: Z_{R1} < Z_{R2} \Rightarrow T(Z_{R1}) < T(Z_{R2})$

{under general topological conditions}

Strategy ordering & its representation

Re1 Arg min $_{\text{any subset of } R^*}$ $T(Z_R)$ is non-dominated

Prop $\boxed{\text{Re1} \Rightarrow Z_{R1} < Z_{R2} \Leftrightarrow T(Z_{R1}) < T(Z_{R2}), \text{ i.e., } T \text{ orders } R^* \text{ completely}}$

{simple contradiction}

Re2 T universal for all orders $<_{Q^*}$ with common N^*, i.e., acts on $Z^*_{R^*} = \cup_{<Q^*} Z_{R^*}$ containing continuous Zs on compact support

T sufficiently smooth and constant preserving

T locally additive, i.e., $T(Z_1+Z_2) = T(Z_1) + T(Z_2)$ for $Z_1 \times Z_2 = 0$

Prop

Re2 $\Rightarrow T(Z) = \int U(Z(Q),Q)\, f(Q)\, dQ$

U utility function shaping the loss in dependence on behavior

$f(Q)$ the pdf describing behavior Q of closed decision loop

{ i) representation of local functional; ii) basic theorem of probability theory, M. Rao “Measure Theory”; iii) existence pdf }

Representation leading to FP DM

Def Let $^{O}R \in \text{Arg min}_{R^*} \int U(Z(Q),Q)\, f(Q)\, dQ$ and denote
$^{I}f(Q) = f(Q)$ for the optimal strategy ^{O}R

U, Z not uniquely determined by the ordering $<_{Q^*}$,
those leading to the same $^{I}f(Q)$ are equivalent

Re3 Representative $W(Z(Q),f(Q)) = U(Z(Q),Q)$ of equivalence class depending on $f(Q)$ smoothly and with $W(Z(Q),{}^{I}f(Q))=\text{constant}$ is searched for

Prop

$$\boxed{\begin{array}{l}\text{Re3} \Rightarrow T(Z) \text{ is affine transformation of} \\ \qquad\qquad \text{the KLD of } f(Q) \text{ on } {}^{I}f(Q) \Leftrightarrow \text{FP DM}\end{array}}$$

{a copy of variation arguments of Bernardo 1978}

Relationships of B DM of FP DM ?

Relationship of FP DM to BDM

Prop $\boxed{\text{To any pair } [L(Q)=U(Z(Q),Q), f(Q)]\ \exists\ {}^{I}f(Q) \Rightarrow \mathrm{E}[L]=\mathrm{D}[f \,\|\, {}^{I}f]}$

$\{\text{Construction } {}^{I}f(Q) = f(Q) \exp[-L(Q)-b(L-E[L])]\}$

Troubles

- Is the inclusion B DM to FP DM legitimate ?
- Generic solution of FP DM randomized
- Optimized functional depends on $f(Q)$ in non-linear way
- Reduction of FP DM on B DM sometimes artificial: FP DM with a set of ideal pdfs is highly desirable

Good news on FP DM

Prop Optimal randomized strategy is given by an explicit formula depending on solution of an integral equation

{Dynamic Programming & elementary properties of KLD}

⇒ Simplicity of the approximated mapping simplifies approximate DP

Prop Ideal pdf can be constructed as a conservative aim-oriented modification of the current closed-loop description

{Find the best reachable *f(Q)* and make a conservative compromise between it and the current one}

⇒ Automatic aim elicitation (… a way towards practice)

⇒ Ideal respects reality and provides robust solutions (… quadratic criteria for heavy-tailed disturbances are non-sense)

Good news on FP DM

Props $\exists$ a rich toolset creating global pdf from low-dimensional pdfs

(unless low-dimensional pdfs are incompatible or conditional)

$\Rightarrow$ FP DM fits to DM with multiple decision makers as

- knowledge sharing $\Leftrightarrow$ creation of global pdf
- aim sharing $\Leftrightarrow$ creation of global ideal pdf

of cooperating neighbors followed by marginalization to respective decision makers

Summarizing (read advertising) statement

FP DM is a rich, relatively new research domain heading to potentially useful practical tool taking us closer to the dreamt DM perpetual motion (a crazy dream, isn't it?) and fully scalable multiple DM