DESIGN OF SQUARE-ROOT DERIVATIVE-FREE SMOOTHERS

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Abstract: Local state estimation approaches for nonlinear stochastic systems are considered. The derivative-free unscented transformation is introduced and used in the design of a local smoothing algorithm. Its numerical properties are discussed and improved by the derivation of the square-root smoothing algorithm. The numerical properties of proposed algorithms are illustrated in a numerical example.

Keywords: stochastic systems, nonlinear systems, state estimation, estimation theory, smoothing

1. INTRODUCTION

The problem of recursive state estimation of discrete-time stochastic dynamic systems from noisy or incomplete measurement data has been a subject of considerable research interest for the last several decades.

The general solution of the estimation problem, based on Bayesian approach, is given by the Functional Recursive Relations (FRR's) for computation of probability density functions (pdf's) of the state conditioned by the measurements. These pdf's provide a full description of the immeasurable state. The FRR's are known for all three parts of the estimation problem which can be distinguished, according to relation between time instant of the estimated state and time instant of the last measurement, to prediction, filtering, and smoothing. It should be mentioned that the FRR's for the filtering and the one-step prediction are known as the Bayesian Recursive Relations (BRR's).

The closed form solution of the FRR's is available only for a few special cases (Lewis, 1986; Šimandl, 1996), e.g. for linear Gaussian system, where the solution of the filtering problem is given by the well-known Kalman Filter. As a solution of smoothing problem, the Rauch-Tung-Striebel Smoother (RTSS) (Lewis, 1986; Šimandl and Královec, 2000) can be used. The alternative approach for smoothing is based on the doubling of the state dimension and on the utilisation of common filtering techniques (Söderström, 1994; Šimandl and Duník, 2006). The multi-step prediction can be imagined as a multiply application of the one-step prediction known from the filtering algorithm (Šimandl and Královec, 2000; Šimandl and Duník, 2006). In other cases it is necessary to apply some approximative methods.

The local methods are often based on approximation of the nonlinear functions in the state or measurement equation so that the Kalman technique can be used for the FRR's solution. This approach causes that all conditional pdf's of the state estimate are given by the first two moments, i.e. mean value and covariance matrix. This rough approximation of the a posteriori estimates induces local validity of the state estimates and consequently impossibility to ensure the convergence of the local filter estimates. Moreover, resulting estimates of the local filters are suitable mainly for point estimates. On the other hand, the advantage of the local methods can be found in the relative simplicity of the FRR's solution.

The standard local nonlinear filtering methods are based on the approximation of nonlinear functions in the state or the measurement equation with the Taylor expansion. The FRR's solution based on the Taylor expansion first order approximation leads to the Extended Kalman Filter (EKF) or to the Iterated Kalman Filter (Lewis, 1986). Generally, the more exact Second Order Filter (Athans *et al.*, 1968; Henriksen, 1982) utilises the Taylor expansion second order approximation. The Taylor expansion first order can be used to design the extended Rauch-Tung-Striebel Smoother and the multi-step predictor as well (Šimandl and Královec, 2000).

In the last decade the novel approaches to the local filter design, based on the polynomial interpolation (Nørgaard *et al.*, 2000; Ito and Xiong, 2000; van der Merwe and Wan, 2001; Duník *et al.*, 2005) or on the unscented transformation (Julier *et al.*, 2000; Ito and Xiong, 2000; Julier and Uhlmann, 2004; van der Merwe and Wan, 2001; Duník *et al.*, 2005), have been published. The approximation of the nonlinear functions by means of the Stirling's polynomial interpolation first or second order leads to the Divide Difference Filters 1st order (DD1) or to the Divided Difference Filter 2nd order (DD2), respectively, which are usually called as the Divided Difference Filters (DDF's) (Nørgaard *et al.*, 2000). Instead of direct substitution of the nonlinear functions in the system description an approximation of the "already approximated" pdf's representing state estimate by a set of deterministically chosen weighted points (so called σ-points) can be utilised as a base for the local filters. This transformation is often called as the unscented transformation. The Unscented Kalman Filter (UKF) (Julier *et al.*, 2000; van der Merwe and Wan, 2001; Šimandl and Duník, 2005) or the Gauss-Hermite Filter (Ito and Xiong, 2000) exemplify this approach. The smoothing local methods utilising the Stirling's interpolation and unscented transformation was very briefly outlined in (Wan and van der Merwe, 2001) and properly derived in (Šimandl and Duník, 2006). Similarly to the standard local approaches the multi-step prediction is realised by the multiple application of the one-step prediction known from the filtering algorithm (Šimandl and Duník, 2006). It is very important to mention that the estimators based on the unscented transformation and the Stirling's interpolation have common features although the basic idea of these estimators comes out from quite different assumptions (Nørgaard *et al.*, 2000; Lefebvre *et al.*, 2002; Duník *et al.*, 2005). Therefore, these local filters can be called together as the sigma point Kalman estimators or the derivative-free Kalman estimators.

The numerical properties of the derivative-free local filters have been discussed in the several papers. For example the DDF's have been directly designed in the square-root form (Nørgaard *et al.*, 2000) and although the UKF was originally derived in the "nonsquare-root" form, its square-root versions have been subsequently derived in (van der Merwe and Wan, 2001; Šimandl and Duník, 2005). However, the poor attention has been paid to the numerical properties of the novel derivative-free smoothers. In (Šimandl and Duník, 2006) only the final algorithms of the derivative-free smoothers were presented without regular their derivation.

Therefore, in the paper the square-root smoothing algorithm, which is based on the unscented

transformation, will be regularly derived. This modification improves not only numerical properties of the smoothing algorithms, but it slightly reduces their computational demands as well. Mention that the same approach presented in this paper can be easily used for design of the smoother based on the both first and second order Stirling's interpolation.

The rest of the paper is organised as follows. Section 2 is devoted to the system description and to the Bayesian solution of the estimation problem. Then, the derivative-free smoother based on the unscented transformation is introduced in Section 3 as well as its square-root modification. In Section 4 the theoretical results are illustrated in a numerical example. Finally, some conclusion remarks are given in Section 5.

2. PROBLEM STATEMENT

Consider the discrete-time nonlinear non-Gaussian stochastic system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{w}_k,\, k = 0, 1, 2, \ldots, \tag{1}$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k,\, k = 0, 1, 2, \ldots, \tag{2}$$

where the vectors $\mathbf{x}_k \in \mathbb{R}^{n_x}$ and $\mathbf{z}_k \in \mathbb{R}^{n_z}$ represent the immeasurable state of the system and the measurement at time instant k, respectively, and $\mathbf{f}_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $\mathbf{h}_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ are known vector functions. The variables $\mathbf{w}_k \in \mathbb{R}^{n_x}$, $\mathbf{v}_k \in \mathbb{R}^{n_z}$ are the state and the measurement Gaussian white noises. The pdf's of both noises $p_{\mathbf{w}_k}(\mathbf{w}_k) = \mathcal{N}\{\mathbf{w}_k : \mathbf{0}_{n_x}, \mathbf{Q}_k\}$, $p_{\mathbf{v}_k}(\mathbf{v}_k) = \mathcal{N}\{\mathbf{v}_k : \mathbf{0}_{n_z}, \mathbf{R}_k\}$ are assumed to be known as well as the pdf of the Gaussian initial state $p_{\mathbf{x}_0}(\mathbf{x}_0)$. The noises are mutually independent and independent of the initial state.

Mention that the system description with non-additive state or measurement noise can be transformed to the previous one (1), (2) by appending noise variables $\mathbf{w}_k$, $\mathbf{v}_k$ to the state $\mathbf{x}_k$ (Nørgaard *et al.*, 2000).

The system can be rewritten as a set of the conditional transient pdf's $p_{\mathbf{x}_{k+1}|\mathbf{x}_k}(\mathbf{x}_{k+1}|\mathbf{x}_k) = p_{\mathbf{w}_k}(\mathbf{x}_{k+1} - \mathbf{f}_k(\mathbf{x}_k))$ and the measurement pdf's $p_{\mathbf{z}_k|\mathbf{x}_k}(\mathbf{z}_k|\mathbf{x}_k) = p_{\mathbf{v}_k}(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k))$, $\forall k$. For the sake of simplicity all pdf's will be given by their arguments, i.e. $p(\mathbf{x}_{k+1}|\mathbf{x}_k) = p_{\mathbf{x}_{k+1}|\mathbf{x}_k}(\mathbf{x}_{k+1}|\mathbf{x}_k)$.

The aim of the state estimation is to find the state estimate in the form of the conditional pdf $p(\mathbf{x}_m|\mathbf{z}^k)$ given by the functional recursive relations known for the prediction ($m > k$), the filtering ($m = k$) and the smoothing ($m < k$). The FRR's are assumed in the following form for the $(m-k)$-step prediction

$$p(\mathbf{x}_m|\mathbf{z}^k) = \int p(\mathbf{x}_m|\mathbf{x}_{m-1})\, p(\mathbf{x}_{m-1}|\mathbf{z}^k) d\mathbf{x}_{m-1}, \tag{3}$$

where $m > k$, for the filtering

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})\, p(\mathbf{z}_k|\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}^{k-1})}, \tag{4}$$

where $p(\mathbf{z}_k|\mathbf{z}^{k-1}) = \int p(\mathbf{x}_k|\mathbf{z}^{k-1})\, p(\mathbf{z}_k|\mathbf{x}_k) d\mathbf{x}_k$, $m = k$, and for the $(k-m)$-step smoothing

$$p(\mathbf{x}_m|\mathbf{z}^k) = p(\mathbf{x}_m|\mathbf{z}^m) \int \frac{p(\mathbf{x}_{m+1}|\mathbf{z}^k)}{p(\mathbf{x}_{m+1}|\mathbf{z}^m)}\, p(\mathbf{x}_{m+1}|\mathbf{x}_m) d\mathbf{x}_{m+1}, \tag{5}$$

where $m < k$ and $\mathbf{z}^k = [\mathbf{z}_0, \ldots, \mathbf{z}_k]$. The recursive computation of the $(m-k)$-step prediction (3) may start either from the filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ or from the predictive pdf $p(\mathbf{x}_t|\mathbf{z}^k)$, where

$m > t > k$. The computation of the filtering pdf (4) comes out from the one-step predictive pdf $p(\mathbf{x}_k|\mathbf{z}^{k-1})$. Finally, the recursion for the smoothed pdf (5) can be initiated either by the filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ or by another smoothed pdf $p(\mathbf{x}_t|\mathbf{z}^k)$, where $m < t < k$.

The exact solution of FRR's (3)–(5) is possible only for a few special cases, e.g. for linear Gaussian systems. In other cases is therefore necessary to apply some approximative method.

The local methods based on the Taylor expansion or on the derivative-free alternatives are known for the prediction, the filtering and the smoothing. While the local predictors and filters are available in a numerical stable versions, the numerical stable versions of the local smoothers have not been properly derived yet. Therefore, the aim of this paper is to propose the technique how to design the numerical stable versions of the smoothers.

3. DESIGN OF DERIVATIVE-FREE SMOOTHERS

The smoothing problem can be generally divided into the three groups, namely fixed-point, fixed-lag and fixed-interval smoothing (Grewal and Andrews, 2001; Šimandl and Duník, 2006), and the local smoothing algorithms based on the Rauch-Tung-Striebel smoother can be easily used for the solution of all three smoothing problem. Therefore, in this paper the main stress is laid on the derivation of numerical stable versions of the local RTSS. However, due the space constrains the stress will be mainly laid on the numerical properties of the derivative-free unscented RTSS.

The algorithms of the local estimators have the same structure as the estimators for the linear Gaussian system, where the predictive, filtering and smoothing means and covariance matrices are computed recursively. The crucial difference between the local estimators can be found in the particular approximation of the system description which allows to apply the techniques known from the linear Gaussian systems.

Therefore, in the first part of this section the Rauch-Tung-Striebel smoother, as base for design of local smoother, will be introduced. Then, the unscented transformation and its application in the smoother design will be recapitulated. Finally, the square-root modification of the unscented smoother will be presented.

3.1 Rauch-Tung-Striebel Smoother

As a main tool for design the unscented smoother, the results from the area of linear system smoother design can be used.

Let the linear Gaussian system (1), (2) where $\mathbf{f}_k(\mathbf{x}_k) = \mathbf{F}_k\mathbf{x}_k$ and $\mathbf{h}_k(\mathbf{x}_k) = \mathbf{H}_k\mathbf{x}_k$, $\forall k$ be considered. For these systems the exact solution of the smoothing problem $p(\mathbf{x}_m|\mathbf{z}^k), m < k$ is given e.g. by the Rauch-Tung-Striebel smoother (RTSS) (Lewis, 1986). The RTSS can be described by the following relations (Lewis, 1986)

$$\hat{\mathbf{x}}_{m|k} = \hat{\mathbf{x}}_{m|m} + \mathbf{K}_{m|k}(\hat{\mathbf{x}}_{m+1|m} - \hat{\mathbf{x}}_{m+1|k}), \tag{6}$$

$$\mathbf{P}_{m|k} = \mathbf{P}_{m|m} - \mathbf{K}_{m|k}(\mathbf{P}_{m+1|m} - \mathbf{P}_{m+1|k})\mathbf{K}_{m|k}^T, \tag{7}$$

$$\mathbf{K}_{m|k} = \mathbf{P}_{xx,m+1|m}\mathbf{P}_{m+1|m}^{-1}, \tag{8}$$

$$\mathbf{P}_{xx,m+1|m} = E[(\mathbf{x}_m - \hat{\mathbf{x}}_m)(\mathbf{x}_{m+1} - \hat{\mathbf{x}}_{m+1|m})^T|\mathbf{z}^m] = \mathbf{P}_{m|m}\mathbf{F}_m^T, \tag{9}$$

where $m = k-1, k-2, \ldots$, $\mathbf{P}_{xx,m+1|m}$ is the cross-covariance matrix of the states $\mathbf{x}_m$ and $\mathbf{x}_{m+1}$, $\hat{\mathbf{x}}_{m|k} = E[\mathbf{x}_m|\mathbf{z}^k]$, $\mathbf{P}_{m|k} = cov[\mathbf{x}_m|\mathbf{z}^k]$ are the smoothed mean and covariance matrix. The filtering mean $\hat{\mathbf{x}}_{m|m} = E[\mathbf{x}_m|\mathbf{z}^m]$ and covariance matrix $\mathbf{P}_{m|m} = cov[\mathbf{x}_m|\mathbf{z}^m]$ and the one-step predictive mean $\hat{\mathbf{x}}_{m|m-1} = E[\mathbf{x}_m|\mathbf{z}^{m-1}]$ and covariance matrix $\mathbf{P}_{m|m-1} = cov[\mathbf{x}_m|\mathbf{z}^{m-1}]$ are known from the "forward" run of the KF.

However, if the nonlinear system is considered, the covariance matrix $\mathbf{P}_{xx,m+1|m}$ (9) cannot be generally computed. To find the solution some approximative technique, e.g. the UT, has to be used. Therefore, in the following section the UT will introduced by an example of the transformation of random variable through nonlinear function.

3.2 Unscented Transformation

Let $\mathbf{x} \in \mathbb{R}^{n_x}$ and $\mathbf{y} \in \mathbb{R}^{n_y}$ be random vector variables related through the known nonlinear function $\mathbf{y} = \mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \ldots, g_{n_y}(\mathbf{x})]^T$. The pdf of the variable $\mathbf{x}$ is characterised by the first two moments, i.e. the mean $\hat{\mathbf{x}}$ and the covariance matrix $\mathbf{P}_x$, and the aim is to calculate the mean $\hat{\mathbf{y}}$ and the covariance matrix $\mathbf{P}_y$ of $\mathbf{y}$ and the cross-covariance matrix $\mathbf{P}_{xy}$. The general solution of this problem is possible only for linear function $\mathbf{g}(\cdot)$. In other cases some approximative solution has to be applied.

One of the possible solution of this "transformation problem" is based on the so called unscented transformation (UT) (Julier *et al.*, 2000). The UT is grounded on the idea that it could be easier to approximate a Gaussian distribution that it is to approximate an arbitrary nonlinear function. Then, the pdf of the random variable $\mathbf{x}$ is approximated by a set of deterministically chosen weighted σ-points $\{\mathcal{X}_i\}$, where

$$\mathcal{X}_0 = \hat{\mathbf{x}}, \mathcal{W}_0 = \frac{\kappa}{n_x + \kappa}, \tag{10}$$

$$\mathcal{X}_i = \hat{\mathbf{x}} + \left(\sqrt{(n_x+\kappa)\mathbf{P}_x}\right)_i, \mathcal{W}_i = \frac{1}{2(n_x+\kappa)}, i = 1, \ldots, n_x, \tag{11}$$

$$\mathcal{X}_j = \hat{\mathbf{x}} - \left(\sqrt{(n_x+\kappa)\mathbf{P}_x}\right)_{j-n_x}, \mathcal{W}_j = \frac{1}{2(n_x+\kappa)}, j = n_x+1, \ldots, 2n_x. \tag{12}$$

The term $\left(\sqrt{(n_x+\kappa)\mathbf{P}_x}\right)_i$ represents i-th column[1] of the matrix $\sqrt{(n_x+\kappa)\mathbf{P}_x}$ and the variable κ is introduced to have a possibility to influence the exactness of the UT. The set of σ-points exactly captures at least the mean and the covariance matrix of $\mathbf{x}$. Then, each point is transformed via the nonlinear function

$$\mathcal{Y}_i = \mathbf{g}(\mathcal{X}_i), \forall i. \tag{13}$$

And the resulting characteristics are given as

$$\hat{\mathbf{y}}_A^{UT} = \sum_{i=0}^{2n_x} \mathcal{W}_i \mathcal{Y}_i, \tag{14}$$

$$\mathbf{P}_{y,A}^{UT} = \sum_{i=0}^{2n_x} \mathcal{W}_i (\mathcal{Y}_i - \hat{\mathbf{y}}_A^{UKF})(\mathcal{Y}_i - \hat{\mathbf{y}}_A^{UKF})^T, \tag{15}$$

$$\mathbf{P}_{xy,A}^{UT} = \sum_{i=0}^{2n_x} \mathcal{W}_i (\mathcal{X}_i - \hat{\mathbf{x}})(\mathcal{Y}_i - \hat{\mathbf{y}}_A^{UKF})^T. \tag{16}$$

[1]The columns are used in the case that $\mathbf{P}_x$ is decomposed as $\mathbf{P}_x = \mathbf{S}_x\mathbf{S}_x^T$. If $\mathbf{P}_x = \mathbf{S}_x^T\mathbf{S}_x$ the rows of $\mathbf{P}_x$ have to be used in the σ-point computation.

The recommended settings of the scaling parameter κ, a tool for adjustment of the UT accuracy, is $\kappa = 3 - n_x$ for the Gaussian distribution (Julier *et al.*, 2000). The introduced subscript A highlights that these results are only approximations of the true mean and the covariance matrix which cannot be generally computed.

3.3 Unscented Rauch-Tung-Striebel Smoother

By combining of the results presented in the previous two subsection, the Unscented Rauch-Tung-Striebel Smoother (URTSS) (Šimandl and Duník, 2006) can be designed. The crucial problem of applying RTSS to the nonlinear system is the impossibility of computation of the cross-covariance matrix $\mathbf{P}_{xx,m+1|m}$ (9). This problem can be now solved by utilising the UT, especially relations (14) and (16). Thus, the approximative solution of $\mathbf{P}_{xx,m+1|m}$ is given by

$$\begin{aligned}\mathbf{P}_{xx,m+1|m} &= E[(\mathbf{x}_m - \hat{\mathbf{x}}_{m|m})(\mathbf{x}_{m+1} - \hat{\mathbf{x}}_{m+1|m})^T|\mathbf{z}^m] \\ &\approx \sum_{i=0}^{2n_x} \mathcal{W}_i(\mathcal{X}_{i,m|m} - \hat{\mathbf{x}}_{m|m})(\mathcal{X}^s_{i,m+1|m} - \hat{\mathbf{x}}^s_{m+1|m})^T, \end{aligned} \tag{17}$$

where $\mathcal{X}^s_{i,m+1|m} = \mathbf{f}_m(\mathcal{X}_{i,m|m}), \forall i$ and $\hat{\mathbf{x}}^s_{m+1|m} = \sum_{i=0}^{2n_x} \mathcal{W}_i \mathcal{X}^s_{i,m+1|m}$ (Šimandl and Duník, 2006). Then, the relations (6)–(8) can be used for obtaining the smoothed characteristics.

The algorithm of the URTSS can be summarised by the following equations

$$\hat{\mathbf{x}}_{m|k} = \hat{\mathbf{x}}_{m|m} + \mathbf{K}_{m|k}(\hat{\mathbf{x}}_{m+1|k} - \hat{\mathbf{x}}_{m+1|m}), \tag{18}$$

$$\mathbf{P}_{m|k} = \mathbf{P}_{m|m} - \mathbf{K}_{m|k}(\mathbf{P}_{m+1|m} - \mathbf{P}_{m+1|k})\mathbf{K}^T_{m|k}, \tag{19}$$

$$\mathbf{K}_{m|k} = \mathbf{P}_{xx,m+1|m}\mathbf{P}^{-1}_{m+1|m}, \tag{20}$$

$$\mathbf{P}_{xx,m+1|m} = \sum_{i=0}^{2n_x} \mathcal{W}_i(\mathcal{X}_{i,m|m} - \hat{\mathbf{x}}_{m|m})(\mathcal{X}^s_{i,m+1|m} - \hat{\mathbf{x}}^s_{m+1|m})^T. \tag{21}$$

It is very important to note, that the square-root of the smoothing covariance matrix $\mathbf{P}_{m|k}$ have to be computed at each time instant to find the appropriate set of the σ-points. Therefore, it could be advantageous to directly compute the square-root of the smoothing covariance matrix. The second, and more important property of the direct computation of the covariance matrix factor, is the significant improvement of the URTSS numerical properties. The square-root URTSS will be proposed in the following part.

Note that in the URTSS design the basic UT was used which suffers from some disadvantages. However for design the URTSS all other improved types of the UT, namely Gauss-Hermite quadrature, scaled UT etc., can be used as well (Ito and Xiong, 2000; Julier and Uhlmann, 2004; Duník *et al.*, 2005).

3.4 Square-Root Unscented Rauch-Tung-Striebel Smoother

To design the square-root form of the URTSS, it is advantageous to define two auxiliary matrices

$$\mathbf{M}_{x,m|m} = [\sqrt{(\mathcal{W}_0)}(\mathcal{X}_{0,m|m} - \hat{\mathbf{x}}_{m|m}), \ldots, \sqrt{(\mathcal{W}_{2n_x})}(\mathcal{X}_{2n_x,m|m} - \hat{\mathbf{x}}_{m|m})], \tag{22}$$

$$\mathbf{M}^s_{x,m+1|m} = [\sqrt{(\mathcal{W}_0)}(\mathcal{X}^s_{0,m+1|m} - \hat{\mathbf{x}}^s_{m+1|m}), \ldots, \sqrt{(\mathcal{W}_{2n_x})}(\mathcal{X}^s_{2n_x,m+1|m} - \hat{\mathbf{x}}^s_{m+1|m})], \tag{23}$$

which facilitate the square-root URTSS design.

Then, the smoothing gain, with respect to (20) and (17), can be written in the form

$$\mathbf{K}_{m|k} = \mathbf{M}_{x,m|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}(\mathbf{P}_{m+1|m})^{-1} = \mathbf{M}_{x,m|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}(\mathbf{S}_{m+1|m}\mathbf{S}^{T}_{m+1|m})^{-1}, \quad (24)$$

where $\mathbf{S}_{m+1|m}$ is the square-root of the predictive covariance matrix $\mathbf{P}_{m+1|m}$. Then, the smoothing covariance matrix (19) can be extended by the following way:

$$\begin{aligned}\mathbf{P}_{m|k} = \mathbf{P}_{m|m} - \mathbf{K}_{m|k}\mathbf{P}_{m+1|m}\mathbf{K}^{T}_{m|k} - \mathbf{K}_{m|k}\mathbf{P}_{m+1|m}\mathbf{K}^{T}_{m|k} + \\ + \mathbf{K}_{m|k}\mathbf{P}_{m+1|m}\mathbf{K}^{T}_{m|k} + \mathbf{K}_{m|k}\mathbf{P}_{m+1|k}\mathbf{K}^{T}_{m|k},\end{aligned} \quad (25)$$

where the term $\mathbf{K}_{m|k}\mathbf{P}_{m+1|m}\mathbf{K}^{T}_{m|k}$ can be expressed as

$$\begin{aligned}\mathbf{K}_{m|k}\mathbf{P}_{m+1|m}\mathbf{K}^{T}_{m|k} &= \mathbf{M}_{x,m|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}\mathbf{K}^{T}_{m|k}, &(26)\\ &= \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}\mathbf{M}^{T}_{x,m|m}, &(27)\\ &= \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}\mathbf{K}^{T}_{m|k} + \mathbf{K}_{m|k}\mathbf{S}_{Q,m}\mathbf{S}^{T}_{Q,m}\mathbf{K}^{T}_{m|k}, &(28)\end{aligned}$$

and $\mathbf{S}_{Q,m}$ is the square-root of the state noise covariance matrix $\mathbf{Q}_{m}$.

The substitution of (26)–(28) into (25) lead to

$$\begin{aligned}\mathbf{P}_{m|k} = \mathbf{M}_{x,m|m}\mathbf{M}^{T}_{x,m|m} - \mathbf{M}_{x,m|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}\mathbf{K}^{T}_{m|k} - \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}\mathbf{M}^{T}_{x,m|m} + \\ + \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}\mathbf{K}^{T}_{m|k} + \mathbf{K}_{m|k}\mathbf{S}_{Q,m}\mathbf{S}^{T}_{Q,m}\mathbf{K}^{T}_{m|k} + \\ + \mathbf{K}_{m|k}\mathbf{S}_{m+1|k}\mathbf{S}^{T}_{m+1|k}\mathbf{K}^{T}_{m|k} = [\mathbf{M}_{x,m|m} - \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}, \mathbf{K}_{m|k}\mathbf{S}_{Q,m}, \mathbf{K}_{m|k}\mathbf{S}_{m+1|k}]\times \\ \times [\mathbf{M}_{x,m|m} - \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}, \mathbf{K}_{m|k}\mathbf{S}_{Q,m}, \mathbf{K}_{m|k}\mathbf{S}_{m+1|k}]^{T}.\end{aligned} \quad (29)$$

Now, the square-root form of the smoothing covariance matrix is given as

$$\tilde{\mathbf{S}}_{m|k} = [\mathbf{M}_{x,m|m} - \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}, \mathbf{K}_{m|k}\mathbf{S}_{Q,m}, \mathbf{K}_{m|k}\mathbf{S}_{m+1|k}]. \quad (30)$$

However, this matrix is rectangular. To compute the set of σ-point the square matrix is desired. Therefore, the Householder triangularization is employed (Grewal and Andrews, 2001; Nørgaard *et al.*, 2000) to transform a known rectangular matrix $\mathbf{M} \in \mathbb{R}^{m\times n}$ to a square matrix $\mathbf{N} \in \mathbb{R}^{n\times n}$ so that the equality $\mathbf{M}\mathbf{M}^{T} = \mathbf{N}\mathbf{N}^{T}$ is accomplished. The Householder triangularization can be written as $\mathbf{N} = ht(\mathbf{M})$.

Now, the "square" square-root form of the smoothing covariance matrix $\mathbf{S}_{m|k}$ can be easily determined, i.e.

$$\mathbf{S}_{m|k} = ht([\mathbf{M}_{x,m|m} - \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}, \mathbf{K}_{m|k}\mathbf{S}_{Q,m}, \mathbf{K}_{m|k}\mathbf{S}_{m+1|k}]). \quad (31)$$

Then, the algorithm of the square-root URTSS is given by the following equations

$$\begin{aligned}\hat{\mathbf{x}}_{m|k} &= \hat{\mathbf{x}}_{m|m} + \mathbf{K}_{m|k}(\hat{\mathbf{x}}_{m+1|k} - \hat{\mathbf{x}}_{m+1|m}), &(32)\\ \mathbf{S}_{m|k} &= ht([\mathbf{M}_{x,m|m} - \mathbf{K}_{m|k}\mathbf{M}^{s}_{x,m+1|m}, \mathbf{K}_{m|k}\mathbf{S}_{Q,m}, \mathbf{K}_{m|k}\mathbf{S}_{m+1|m}]), &(33)\\ \mathbf{K}_{m|k} &= \mathbf{P}_{xx,m+1|m}(\mathbf{S}_{m+1|m}\mathbf{S}^{T}_{m+1|m})^{-1}, &(34)\\ \mathbf{P}'_{xx,k+1} &= \mathbf{M}_{x,m|m}(\mathbf{M}^{s}_{x,m+1|m})^{T}. &(35)\end{aligned}$$

Algorithm	MSE of fixed-lag smoothing	Time [s]
ERTSS	1.36×10^{-3}	1.51^{-4}
URTSS	1.25×10^{-3}	3.27^{-4}
sURTSS	1.25×10^{-3}	2.76^{-4}

Table 1: MSE of fixed-lag smoothing and computational demands.

For the design of the square-root derivative-free smoother the Stirling's interpolation (Nørgaard *et al.*, 2000; Šimandl and Duník, 2006) can be used as well. Due to the common properties of Stirling's interpolation and the UT the resultant relations are quite similar to the above presented.

4. NUMERICAL ILLUSTRATION

Let the one-dimensional non-linear Gaussian system (Ito and Xiong, 2000; Šimandl and Duník, 2006) be considered

$$x_{k+1} = x_k + 5\Delta t x_k(1 - x_k^2) + w_k, \tag{36}$$

$$z_k = \Delta t(x_k - 0.05)^2 + v_k, \tag{37}$$

where $k = 0, 1, \ldots, 400$, system initial condition $x_0 = 1.2$, $w_k \sim \mathcal{N}\{w_k : 0, 0.25\Delta t\}$, $v_k \sim \mathcal{N}\{w_k : 0, 0.01\Delta t\}$ and $\Delta t = 0.01$. Initial condition of the estimators is considered as $p(x_0|z^{-1}) = \mathcal{N}\{x_0 : 2.2, 2\}$.

The aim is to find the two-step fixed-lag smoothing estimates $p(x_{k-2}|z^k)$ by means of the standard local RTSS based on the Taylor expansion 1st order (Extended RTSS - ERTSS), the novel Unscented RTSS and the square-root URTSS (sURTSS). The experimental results are summarised in Table 1 where the Mean Square Error (MSE)[2] of the fixed-lag smoothed estimate and the computational time per one smoother run are given.

The improvement of estimation performance of URTSS towards the Extended RTSS is proportional to the improvement of the UKF towards the EKF. Mention that the improvement becomes significant especially for the "highly" nonlinear systems. Naturally, the estimation performance is the same for both URTSS. The square-root URTSS brings the smaller computational demands especially for the low dimensional systems. However, the main advantage of the square-root smoother can be found in the improved numerical stability where the smoothed covariance matrix can not be negative-definite.

5. CONCLUSION

The derivative-free smoothing methods were discussed. The unscented transformation, as a derivative-free approximation technique, was briefly discussed and used in the design of the Unscented Rauch-Tung-Striebel Smoother. The square-root version of the Unscented Rauch-Tung-Striebel Smoother was derived to improve the numerical properties of that algorithm. The proposed method for design square-root smoothers can be easily used for other smoothers that are based on different approximation techniques, e.g. Stirling's interpolation, Gauss-Hermite quadrature, scaled unscented transformation. The proposed smoother was demonstrated on the numerical example.

[2] $MSE = \left(\sum_{i=1}^{100}\sum_{k=0}^{400}(x_k^i - \hat{x}_k^{i,E})^2\right)/(401 \times 100)$, where x_k^i or $\hat{x}_k^{i,E}$ is the true or estimated state, respectively, in the i-th repetition and N is the time instant of the last possible estimate.

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REFERENCES

Athans, M., R. P. Wisher and A. Bertolini (1968). Suboptimal state estimation for continuous-time nonlinear systems with discrete noisy measurements. *IEEE Transactions On Automatic Control* (5), 504–514.

Duník, J., M. Šimandl, O. Straka and L. Král (2005). Performance analysis of derivative-free filters. In: *Proceedings of the 44th IEEE Conference on Decision and Control, and European Control Conference ECC'05*. number ISBN: 0-7803-9568-9, ISSN: 0191-2216. Seville, Spain. pp. 1941–1946.

Grewal, S. G. and A. P. Andrews (2001). *Kalman Filtering: Theory and Practise Using MATLAB (Second Edition)*. John Wiley & Sons. New York. ISBN 0-471-39254-5.

Henriksen, R. (1982). The truncated second-order nonlinear filter revised. *IEEE Transactions On Automatic Control* **27**(1), 247–251.

Ito, K. and K. Xiong (2000). Gaussian filters for nonlinear filtering problems. *Automatica* **45**(5), 910–927.

Julier, S. J. and J. K. Uhlmann (2004). Unscented filtering and nonlinear estimation. *IEEE review* **92**(3), 401–421.

Julier, S. J., J. K. Uhlmann and H. F. Durrant-White (2000). A new method for the nonlinear transformation of means and covariances in filters and estimators. *IEEE Transactions On Automatic Control* **45**(3), 477–482.

Lefebvre, T., H. Bruyninckx and J. De Schuller (2002). Comment on "a new method for the nonlinear transformation of means and covariances in filters and estimators". *IEEE Transactions On Automatic Control* **47**(8), 1406–1409.

Lewis, F. L. (1986). *Optimal Estimation*. John Wiley & Sons. New York. ISBN 0-471-83741-5.

Nørgaard, M., N. K. Poulsen and O. Ravn (2000). New developments in state estimation for nonlinear systems. *Automatica* **36**(11), 1627–1638.

Söderström, T. (1994). Discrete-time stochastic systems. In: *ser. Systems and Control Engineering*. UpperSaddle River, NJ: Prentice-Hall.

van der Merwe, R. and E. A. Wan (2001). The square-root unscented kalman filter for state and parameter-estimation. In: *International Conference on Acoustics, Speech, and Signal Processing*. Salt Lake City, Utah.

Šimandl, M. (1996). State estimation for non-gaussian models. In: *Preprints of the 13Th World Congress*. Vol. 2. IFAC. Elsevier Science. San Francisco, USA. pp. 463–468.

Šimandl, M. and J. Duník (2005). Sigma point gaussian sum filter design using square root unscented filters. In: *Preprints of 16th IFAC World Congress*. Prague, Czech Republic.

Šimandl, M. and J. Duník (2006). Design of derivative-free smoothers and predictors. In: *Preprints of the 14th IFAC Symposium on System Identification*. Newcastle, Australia.

Šimandl, M. and J. Královec (2000). Filtering, prediction and smoothing with gaussian sum representation. In: *Proceedings of the 12th IFAC Symposium on System Identification*. Santa Barbara, California, USA.

Wan, E. A. and van der Merwe, R., Eds. (2001). *Kalman Filtering and Neural Networks*. pp. 221–269. Wiley Publishing.