# A feasible design of active detector and input signal generator

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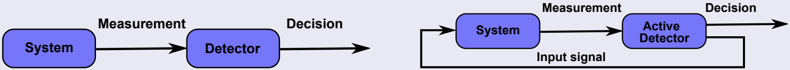
# Outline

# Introduction

## Active detection – authors' earlier results




- General formulation of active detection and control problem
- Formal solution to this problem
- Included interesting special cases
    1. Design of active detector for given input signal generator
    2. Design of active detector and input signal generator
    3. Design of active detector and controller
- Feasible suboptimal designs (receding/rolling horizon technique)

# Introduction

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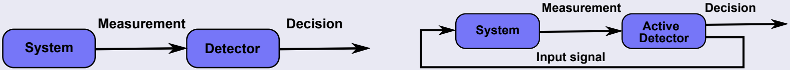


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# Introduction – cont'd

## Goals of presentation

- Design optimal active detector and input signal generator for jump Markov linear Gaussian models
- Derive optimal active detector and input signal generator for discrimination between several linear Gaussian models
- Discuss two suboptimal design techniques

# Problem Formulation

## Stochastic discrete-time system at time $k \in \mathcal{T} = \{0, 1, \ldots, F\}$

$$\mathbf{x}_{k+1} = \mathbf{A}_{\mu_k}\mathbf{x}_k + \mathbf{B}_{\mu_k}\mathbf{u}_k + \mathbf{G}_{\mu_k}\mathbf{w}_k$$
$$\mathbf{y}_k = \mathbf{C}_{\mu_k}\mathbf{x}_k + \mathbf{H}_{\mu_k}\mathbf{v}_k$$

$\left[\mathbf{x}_k^T, \mu_k\right]^T$ – system state, $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\mu_k \in \mathcal{M} = \{1, \ldots, N\}$
$\mathbf{u}_k \in \mathcal{U}_k \subseteq \mathbb{R}^{n_u}$ – input, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ – output
$\mathcal{N}\{\mathbf{w}_k : \mathbf{0}, \mathbf{I}\}$ – state noise, $\mathcal{N}\{\mathbf{v}_k : \mathbf{0}, \mathbf{I}\}$ – output noise
$\mathcal{N}\{\mathbf{x}_0 : \hat{\mathbf{x}}_0', \mathbf{P}_0'\}$ – initial condition
$P_{i,j} = P(\mu_{k+1} = j | \mu_k = i)$ – transition probabilities
$P(\mu_0)$ – initial condition

# Problem Formulation – cont'd

## Active detector and input signal generator at time $k \in \mathcal{T}$

$$\begin{bmatrix} d_k \\ \mathbf{u}_k \end{bmatrix} = \boldsymbol{\rho}_k \left( \mathbf{I}_0^k \right)$$

$d_k \in \mathcal{M}$ – decision (point estimate of $\mu_k$)
$\boldsymbol{\rho}_k(\cdot)$ – unknown generally nonlinear function
$\mathbf{I}_0^k = \left[ \mathbf{y}_0^k, \mathbf{u}_0^{k-1}, d_0^{k-1} \right]$ – information received up to time $k$

# Problem Formulation – cont'd

## Design goal

The aim is to design an active detector and input signal generator (i.e. sequence of functions $\rho_k$) that minimizes average losses/costs caused by wrong decisions.

## Criterion

$$J\left(\rho_0^F\right) = \mathsf{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{d}}(\mu_k, d_k)\right\} \to \min$$

$L_k^{\mathrm{d}}(\mu_k, d_k)$ – loss (cost) function that satisfies condition
$L_k^{\mathrm{d}}(\mu_k, \mu_k) \leq L_k^{\mathrm{d}}(\mu_k, d_k),\ \forall d_k, \mu_k \in \mathcal{M},\ d_k \neq \mu_k,\ \exists k \ \leq \to <$

# Optimal Design

## Backward recursive equation

$$V_k^*\left(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}\right) = \min_{d_k \in \mathcal{M}} \mathrm{E}\left\{ L_k^{\mathrm{d}}\left(\mu_k, d_k\right) | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}, d_k \right\} + \min_{\mathbf{u}_k \in \mathcal{U}_k} \mathrm{E}\left\{ V_{k+1}^*\left(\mathbf{y}_0^{k+1}, \mathbf{u}_0^k\right) | \mathbf{y}_0^k, \mathbf{u}_0^k \right\}$$

$V_k^*\left(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}\right)$ – Bellman's function (expected costs)
$V_{F+1}^* = 0$ – initial condition for recursion
$J\left(\rho_0^{F*}\right) = \mathrm{E}\left\{V_0^*\left(\mathbf{y}_0\right)\right\}$ – optimal value of the criterion

# Optimal Design – cont'd

## Optimal active detector and input signal generator

$$\begin{bmatrix} d_k^* \\ \mathbf{u}_k^* \end{bmatrix} = \rho_k^* \left( \mathbf{I}_0^k \right) = \begin{bmatrix} \arg \min\limits_{d_k \in \mathcal{M}} \mathrm{E} \left\{ L_k^{\mathrm{d}} \left( \mu_k, d_k \right) | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}, d_k \right\} \\ \arg \min\limits_{\mathbf{u}_k \in \mathcal{U}_k} \mathrm{E} \left\{ V_{k+1}^* \left( \mathbf{y}_0^{k+1}, \mathbf{u}_0^k \right) | \mathbf{y}_0^k, \mathbf{u}_0^k \right\} \end{bmatrix}$$

The optimal decision $d_k^*$ minimizes current average costs, and the optimal input $\mathbf{u}_k^*$ minimizes future average costs.

# Problem of Model Discrimination

## Assumptions

- Original formulation applies
- Additional assumptions
  - There is no switching between models

$$P_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

  - Only the terminal decision $d_F$ is of interest

$$L_k^{\mathrm{d}}(\mu_k, d_k) = \begin{cases} 1 & \text{if } d_k \neq \mu_k \wedge k = F, \\ 0 & \text{otherwise.} \end{cases}$$

# Optimal Design for Model Discrimination

## Backward recursive equation at time step $k = F$

$$V_F^*\left(\mathbf{y}_0^F, \mathbf{u}_0^{F-1}\right) = \min_{d_F \in \mathcal{M}} \mathrm{E}\left\{L_F^{\mathrm{d}}(\mu_F, d_F) \,|\mathbf{y}_0^F, \mathbf{u}_0^{F-1}, d_F\right\} + \underbrace{\min_{\mathbf{u}_F \in \mathcal{U}_F} \mathrm{E}\left\{V_{F+1}^*\left(\mathbf{y}_0^{F+1}, \mathbf{u}_0^F\right) |\mathbf{y}_0^F, \mathbf{u}_0^F\right\}}_{0}$$

# Optimal Design for Model Discrimination

**Backward recursive equation at time step $k = F$**

$$V_F^*\left(\mathbf{y}_0^F, \mathbf{u}_0^{F-1}\right) = \min_{d_F \in \mathcal{M}} \mathrm{E}\left\{L_F^{\mathrm{d}}\left(\mu_F, d_F\right) \left| \mathbf{y}_0^F, \mathbf{u}_0^{F-1}, d_F \right.\right\}$$

$$d_F^* = \arg \min_{d_F \in \mathcal{M}} \mathrm{E}\left\{L_F^{\mathrm{d}}\left(\mu_F, d_F\right) \left| \mathbf{y}_0^F, \mathbf{u}_0^{F-1}, d_F \right.\right\}$$

$$\mathbf{u}_F^* = ?$$

According to the chosen cost function $L_F^{\mathrm{d}}\left(\mu_F, d_F\right)$, the optimal decision $d_F^*$ corresponds to a model with maximum a posteriori probability. The optimal input $\mathbf{u}_F^*$ can be pick out arbitrarily.

# Optimal Design for Model Discrimination

**Backward recursive equation at time steps $k = F-1, \dots, 0$**

$$V_k^*\left(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}\right) = \overbrace{\min_{d_k \in \mathcal{M}} \mathrm{E}\left\{ L_k^{\mathrm{d}}\left(\mu_k, d_k\right) | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}, d_k \right\}}^{0} + \min_{\mathbf{u}_k \in \mathcal{U}_k} \mathrm{E}\left\{ V_{k+1}^*\left(\mathbf{y}_0^{k+1}, \mathbf{u}_0^k\right) | \mathbf{y}_0^k, \mathbf{u}_0^k \right\}$$

# Optimal Design for Model Discrimination

**Backward recursive equation at time steps $k = F-1, \dots, 0$**

$$V_k^*\left(\mathbf{y}_0^k, \mathbf{u}_0^{k-1}\right) = \min_{\mathbf{u}_k \in \mathcal{U}_k} \mathrm{E}\left\{ V_{k+1}^*\left(\mathbf{y}_0^{k+1}, \mathbf{u}_0^k\right) |\mathbf{y}_0^k, \mathbf{u}_0^k \right\}$$

$$d_k^* = ?$$

$$\mathbf{u}_k^* = \arg\min_{\mathbf{u}_k \in \mathcal{U}_k} \mathrm{E}\left\{ V_{k+1}^*\left(\mathbf{y}_0^{k+1}, \mathbf{u}_0^k\right) |\mathbf{y}_0^k, \mathbf{u}_0^k \right\}$$

The optimal decision $d_k^*$ can be chosen arbitrarily, but the optimal input $\mathbf{u}_k^*$ minimizes expected future average costs.

# Suboptimal Designs for Model Discrimination

## Suboptimal design techniques

1. Design based on upper bound for Bayesian risk
   - Interchange of minimization and expectation operators
   - Replacement of Bayesian risk by its upper bound
2. Design based on $\ell$-step rollout policy
   - Numerical computation of Bellman's functions over $\ell$ steps
   - Replacement of true Bellman's function $V^*_{k+l}\left(\mathbf{y}_0^{k+l}, \mathbf{u}_0^{k+l-1}\right)$ by a function $\bar{V}_{k+l}\left(\mathbf{y}_0^{k+l}, \mathbf{u}_0^{k+l-1}\right)$ computed using a suboptimal policy (the so called base policy)

# Design Based on Upper Bound for Bayesian Risk

## Interchange of minimization and expectation operators

- Subsequent substitution of Bellman's functions
  $V_F^* \to \ldots \to V_k^* \to \ldots \to V_1^* \to V_0^*$

$$V_0^*(\mathbf{y}_0) = \min_{\mathbf{u}_0 \in \mathcal{U}_0} \mathrm{E}\left\{ \min_{\mathbf{u}_1 \in \mathcal{U}_1} \mathrm{E}\left\{ \ldots \min_{\mathbf{u}_{F-1} \in \mathcal{U}_{F-1}} \mathrm{E}\left\{\right\} \ldots | \mathbf{y}_0^1, \mathbf{u}_0^1 \right\} | \mathbf{y}_0, \mathbf{u}_0 \right\}$$

- Interchange of minimization and expectation operators

$$V_0^*(\mathbf{y}_0) \leq \min_{\mathbf{u}_0^{F-1}} \overbrace{\int_{\mathbb{R}^{Fn_y}} \min_{d_F \in \mathcal{M}} \sum_{\substack{\mu_F = 1 \\ \mu_F \neq d_F}}^{N} p\left(\mathbf{y}_1^F | \mathbf{y}_0, \mathbf{u}_0^{F-1}, \mu_F\right) P\left(\mu_F | \mathbf{y}_0\right) \mathrm{d}\,\mathbf{y}_1^F}^{\text{Bayesian risk } \epsilon\left(\mathbf{y}_0, \mathbf{u}_0^{F-1}\right)}$$

# Design Based on Upper Bound for Bayesian Risk

## Implications of interchanging min{·} and E{·} operators

- "Simple" static optimization problem need to be solved
- Inputs are designed using Open Loop Strategy
- The problem considered in [*Blackmore, L. and Williams, B.C. (2006). Finite horizon control design for optimal discrimination between several models.*] is obtained

## Notes on Bayesian risk

- It expresses a priori probability of a wrong decision
- Distributions $p\left(\mathbf{y}_1^F | \mathbf{y}_0, \mathbf{u}_0^{F-1}, \mu_F\right)$ are Gaussian
- It cannot be evaluated analytically

# Design Based on Upper Bound for Bayesian Risk

## Upper bound for Bayesian risk $\epsilon(\mathbf{y}_0, \mathbf{u}_0^{k-1})$

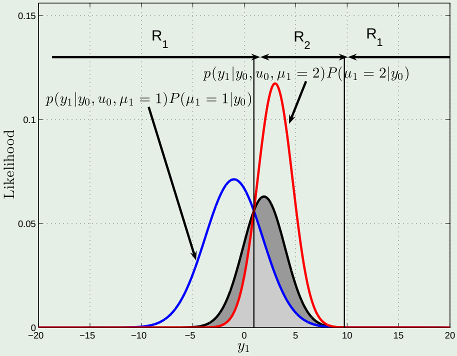
$$\epsilon\left(\mathbf{y}_0, \mathbf{u}_0^{F-1}\right) \leq \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sqrt{P\left(\mu_F = i|\mathbf{y}_0\right) P\left(\mu_F = j|\mathbf{y}_0\right)}\, \mathrm{e}^{-D_{i,j}}$$

$D_{i,j}$ – Bhattacharrya distance between $p\left(\mathbf{y}_1^F|\mathbf{y}_0, \mathbf{u}_0^{F-1}, \mu_F = i\right)$ and $p\left(\mathbf{y}_1^F|\mathbf{y}_0, \mathbf{u}_0^{F-1}, \mu_F = j\right)$

- The function $D_{i,j}$ is a convex quadratic function of $\mathbf{u}_0^{F-1}$
- Numerical constrained optimization can be used

# Design Based on Upper Bound for Bayesian Risk

## Illustration of upper bound and Bayesian risk for two models



# Design based on $\ell$-step rollout policy

## Numerical computation of Bellman's functions over $\ell$-steps

- The space of inputs is discretized for continuous sets $\mathcal{U}_k$
- The criterion value is computed for each point of the grid
- The optimal input is found by criterion values comparison

## Replacement of the true Bellman's function $V^*_{k+l}(\mathbf{y}_0^{k+l}, \mathbf{u}_0^{k+l-1})$

$$V^*_{k+l}(\cdot) \approx \bar{V}_{k+l}(\cdot) = \min_{\mathbf{u}_{k+l}^{F-1}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathrm{e}^{-D_{i,j}} \sqrt{P(\mu_F = i|\cdot)\, P(\mu_F = j|\cdot)}$$

# Numerical Example

## Discrimination between two models

| $\mu_k$ | $a_{\mu_k}$ | $b_{\mu_k}$ | $g_{\mu_k}$ | $c_{\mu_k}$ | $h_{\mu_k}$ |
|---|---|---|---|---|---|
| 1 | -0.9 | 1.0 | $\sqrt{0.5}$ | 1.5 | $\sqrt{0.5}$ |
| 2 | 0.5 | 1.5 | $\sqrt{0.5}$ | 1.5 | $\sqrt{0.5}$ |

- Detection horizon $F = 2$
- Initial conditions $P(\mu_0 = 1) = P(\mu_0 = 2) = 0.5$, $\hat{\mathbf{x}}'_0 = 0$ and $\mathbf{P}'_{0,x} = 0.1$
- Set of admissible input $\mathcal{U}_k = \{-0.1, 0.1\}$, $\forall k \in \mathcal{T}$
- Length of optimization horizon $\ell = 1$

# Numerical Example – cont'd

## Result of 1000 Monte Carlo simulations

| Design method | $\hat{J}$ | $\operatorname{var} \hat{J}$ | Time [s] |
|---|---|---|---|
| Optimal design | 0.3300 | 2.4380E-4 | 53.4573 |
| Rollout policy | 0.3350 | 2.4116E-4 | 1.1104 |
| Bayesian risk (BR) | 0.3380 | 2.3843E-4 | 6.0872 |
| Upper bound for BR | 0.3430 | 2.2756E-4 | 0.0085 |

# Concluding Remarks

- Both suboptimal designs were derived from the optimal design by taking some approximations
- It was shown that the approach proposed by *L. Blackmore and B.C. Williams* can derived from the optimal design using two subsequent approximations
- The quality of detection can be improved using the $\ell$-step rollout policy