Multistage Prediction Error Adaptive Dual Controller

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Outline

Dual adaptive control

- Control problem with unknown state and parameters
- Two conflicting goals – meet control objective and improve estimation
- Aspects of dual control
  - **Caution** - due to inherent uncertainties
  - **Probing (Active learning)** - helps decrease the uncertainty about the unknown state and parameters
- Optimal adaptive dual control problem – mostly cannot be solved analytically

Suboptimal solutions

- with constraint to one-step control horizon
  - Augmenting the cautious control law (Bicriterial controller,. . .)
  - Modification of criterion (e.g. PEDC, IDC, ASOD,. . .)
- with two- or multiple step control horizon
  - Criterion approximation (e.g. WDC, Utility cost,. . .)

Goal: to find feasible solution

Requirements of feasible solution

- ✔ computationally moderate not only for one step ahead horizon
- ✔ clear interpretation
- ✔ guarantees sufficient control quality

Deficiencies of current approaches

- ❢ either limited to one step ahead horizon or computationally demanding

Steps to fulfil the goal

1. formulation of optimisation problem with arbitrary control horizon
2. choice of probability density function approximation the would make possible to find closed form solution.
3. assurance of both properties of the dual control

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Considered system

$$\boldsymbol{s}_{k+1} = \boldsymbol{A}(\boldsymbol{\theta}_k)\boldsymbol{s}_k + \boldsymbol{B}(\boldsymbol{\theta}_k)\boldsymbol{u}_k + \boldsymbol{w}_k, \tag{1}$$

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\Phi}_k\boldsymbol{\theta}_k + \boldsymbol{\epsilon}_k, \qquad k = 0, \ldots, N-1 \tag{2}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{s}_k + \boldsymbol{v}_k, \tag{3}$$

$\boldsymbol{s}_k \in \mathbb{R}^n$	...	non-measurable state
$\boldsymbol{\theta}_k \in \mathbb{R}^p$	...	unknown parameters
$\boldsymbol{u}_k \in \mathbb{R}^r$	...	control
$\boldsymbol{y}_k \in \mathbb{R}^m$	...	measurement

- ✓ The elements of matrices $\boldsymbol{A}(\boldsymbol{\theta}_k)$ and $\boldsymbol{B}(\boldsymbol{\theta}_k)$ are known linear function of the unknown parameters $\boldsymbol{\theta}_k$.
- ✓ The random quantities $\boldsymbol{s}_0$, $\boldsymbol{\theta}_0$, $\boldsymbol{w}_k$, $\boldsymbol{\epsilon}_k$ and $\boldsymbol{v}_k$ are described by known pdf's and are mutually independent.

Optimisation problem

General optimisation problem

The aim is to find control law

$$\boldsymbol{u}_k = \boldsymbol{u}_k(\boldsymbol{I}_k) = \boldsymbol{u}_k(\boldsymbol{u}_0^{k-1}, \boldsymbol{y}_0^k), \qquad k = 0, 1, \ldots, N-1$$

that minimises the following criterion

$$J = E\left\{\mathcal{L}(\boldsymbol{u}_0^{N-1}, \boldsymbol{s}_0^{N-1}, \boldsymbol{\theta}_0^{N-1})\right\}$$

with respect to the system (1)-(3).

Common choice of the cost function $\mathcal{L}(\boldsymbol{u}_0^{N-1}, \boldsymbol{s}_0^{N-1}, \boldsymbol{\theta}_0^{N-1})$

$$\mathcal{L}(\boldsymbol{u}_0^{N-1}, \boldsymbol{s}_0^{N-1}, \boldsymbol{\theta}_0^{N-1}) = \sum_{k=0}^{N-1} (\boldsymbol{s}_{k+1} - \bar{\boldsymbol{s}}_{k+1})^T \boldsymbol{Q}_{k+1} (\boldsymbol{s}_{k+1} - \bar{\boldsymbol{s}}_{k+1}) + \boldsymbol{u}_k^T \boldsymbol{R}_k \boldsymbol{u}_k$$

Optimisation problem

Solvability of the optimisation problem

- general solution given by Bellman optimisation recursion
- analytically unsolvable (due to inherent nonlinearities)
- it is necessary to use some approximation

Possible approximation choices

- Enforced Certainty equivalence $\rightarrow$ leads to LQG controller

$$\rho_k^{CE} = \Big\{ p(\boldsymbol{s}_{k+i}, \boldsymbol{\theta}_{k+i} | \boldsymbol{I}_{k+i}) \simeq \delta(\boldsymbol{s}_{k+i} - \hat{\boldsymbol{s}}_{k+i}) \delta(\boldsymbol{\theta}_{k+i} - \hat{\boldsymbol{\theta}}_{k+i|k}); \quad i = 0, \ldots, N-k-1 \Big\}$$

- Partial Certainty equivalence (PCE)

$$\rho_k = \Big\{ p(\boldsymbol{s}_{k+i}, \boldsymbol{\theta}_{k+i} | \boldsymbol{I}_{k+i}) \simeq \delta(\boldsymbol{s}_{k+i} - \hat{\boldsymbol{s}}_{k+i}) p(\boldsymbol{\theta}_{k+i} | \boldsymbol{I}_k); \quad i = 0, \ldots, N-k-1 \Big\} \tag{4}$$

Optimisation problem

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Reformulation of the optimisation problem

Reformulated optimisation problem employing PCE approximation

Control law sought as to minimise the criterion

$$J = E_{\rho_0}\left\{\mathcal{L}(\boldsymbol{u}_0^{N-1}, \boldsymbol{s}_0^{N-1}, \boldsymbol{\theta}_0^{N-1})\right\}$$

- the expectations determined using ρ approximation (4)
- the control law is suboptimal with respect to original formulation
- not strictly *closed-loop* anymore

Adaptive control based on PCE approximation

$$\boldsymbol{u}_k = \underset{u_k}{\operatorname{argmin}}\, J_k\left(\boldsymbol{I}_k\right), \qquad k = 0, 1, \ldots, N-1$$

$$J_k\left(\boldsymbol{I}_k\right) = E_{\rho_k}\left\{\mathcal{L}(\boldsymbol{u}_k^{N-1}, \boldsymbol{s}_k^{N-1}, \boldsymbol{\theta}_k^{N-1})\middle|\boldsymbol{I}_k\right\} = E_{\rho_k}\left\{\sum_{i=k}^{N-1}\mathcal{L}_i\left(\boldsymbol{s}_i, \boldsymbol{\theta}_i, \boldsymbol{u}_i\right)\middle|\boldsymbol{I}_k\right\}$$

$$\mathcal{L}_i\left(\boldsymbol{s}_i, \boldsymbol{\theta}_i, \boldsymbol{u}_i\right) = \left(\boldsymbol{s}_{i+1} - \bar{\boldsymbol{s}}_{i+1}\right)^T \boldsymbol{Q}_{i+1}\left(\boldsymbol{s}_{i+1} - \bar{\boldsymbol{s}}_{i+1}\right) + \boldsymbol{u}_i^T \boldsymbol{R}_i \boldsymbol{u}_i$$

This controller is of cautious type, i.e. it isn't dual controller!

Reformulation of the optimisation problem

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The PCE approximation ensures only cautious behaviour

❢ It is necessary to modify the criterion

Useful criterion modification

Cost function used in **Prediction Error Dual Controller (PEDC)**:

$$\mathcal{L}_i(\cdot) = (\boldsymbol{s}_{k+1} - \bar{\boldsymbol{s}}_{k+1})^T \boldsymbol{Q}_{k+1} (\boldsymbol{s}_{k+1} - \bar{\boldsymbol{s}}_{k+1}) + \boldsymbol{u}_k^T \boldsymbol{R}_k \boldsymbol{u}_k - \boldsymbol{\nu}_{k+1}^T \boldsymbol{\Lambda}_{k+1} \boldsymbol{\nu}_{k+1}$$

- ✔ simple cost function modification with clear interpretation
- ✔ the quality of estimates rated using prediction error
- ✔ the degree of compromise tuned independently for each parameter
- ✔ still analytically solvable using PCE

Multi-Stage Prediction Error Dual Controller (MSPEDC)

The modified control objective criterion

$$J_k = E_{\rho_k}\left\{\sum_{i=k}^{N-1} \left(\boldsymbol{s}_{i+1} - \bar{\boldsymbol{s}}_{i+1}\right)^T \boldsymbol{Q}_{i+1} \left(\boldsymbol{s}_{i+1} - \bar{\boldsymbol{s}}_{i+1}\right) + \boldsymbol{u}_i^T \boldsymbol{R}_i \boldsymbol{u}_i - \boldsymbol{v}_{i+1}^T \boldsymbol{\Lambda}_{i+1} \boldsymbol{v}_{i+1} \middle| \boldsymbol{I}_k\right\}$$

where

$$\boldsymbol{v}_{i+1} = \boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}(\hat{\boldsymbol{s}}_i, \hat{\boldsymbol{\theta}}_{i|k}),\ \boldsymbol{x}_i \triangleq \begin{pmatrix} \boldsymbol{s}_i \\ \boldsymbol{\theta}_i \end{pmatrix},\ \hat{\boldsymbol{x}}_{i+1|i} \triangleq E_{\rho_k}\left\{\boldsymbol{x}_{i+1} \middle| \boldsymbol{I}_i\right\} = \begin{pmatrix} \hat{\boldsymbol{s}}_{i+1|i} \\ \hat{\boldsymbol{\theta}}_{i+1|k} \end{pmatrix}$$

and the prediction of the augmented state $\hat{\boldsymbol{x}}_{i+1|i}$ is defined as

$$\hat{\boldsymbol{x}}_{i+1|i} = \begin{pmatrix} \boldsymbol{A}(\hat{\boldsymbol{\theta}}_{i|k}) & \mathcal{O} \\ \mathcal{O} & \boldsymbol{\Phi}_i \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{s}}_i \\ \hat{\boldsymbol{\theta}}_{i|k} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}(\hat{\boldsymbol{\theta}}_{i|k}) \\ \mathcal{O} \end{pmatrix} \boldsymbol{u}_i + \begin{pmatrix} \hat{\boldsymbol{w}}_i \\ \hat{\boldsymbol{\epsilon}}_i \end{pmatrix}.$$

with

$$\hat{\boldsymbol{s}}_i \triangleq E_{\rho_k}\left\{\boldsymbol{s}_i \middle| \boldsymbol{I}_i\right\},\quad \hat{\boldsymbol{\theta}}_{i|k} \triangleq E_{\rho_k}\left\{\boldsymbol{\theta}_i \middle| \boldsymbol{I}_k\right\},\quad \hat{\boldsymbol{w}}_i \triangleq E\left\{\boldsymbol{w}_i\right\},\quad \hat{\boldsymbol{\epsilon}}_i \triangleq E\left\{\boldsymbol{\epsilon}_i\right\}.$$

Analysis of the criterion

Decomposition of the criterion

$$J_k = J_k^{\mathcal{C}} + J_k^{\mathcal{P}}$$

$\Rightarrow$ it comprises both aspect of the dual control

- Cautious part (it's equivalent to the original quadratic criterion)

$$J_k^{\mathcal{C}} = \sum_{i=k}^{N-1} \left(\hat{\boldsymbol{s}}_{i+1|i} - \bar{\boldsymbol{s}}_{i+1}\right)^T \boldsymbol{Q}_{i+1} \left(\hat{\boldsymbol{s}}_{i+1|i} - \bar{\boldsymbol{s}}_{i+1}\right) + \boldsymbol{u}_i^T \boldsymbol{R}_i \boldsymbol{u}_i +$$
$$+ E_{\rho_k} \left\{ \sum_{i=k}^{N-1} \left(\boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}\right)^T \boldsymbol{V}_{i+1} \left(\boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}\right) \middle| \boldsymbol{I}_k \right\}$$

- Probing part

$$J_k^{\mathcal{P}} = -E_{\rho_k} \left\{ \sum_{i=k}^{N-1} \left(\boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}\right)^T \boldsymbol{\Lambda}_{i+1} \left(\boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}\right) \middle| \boldsymbol{I}_k \right\}$$

The solution of the modified optimisation problem

Bellman optimisation recursion

$$\mathcal{V}_i^o = \min_{\boldsymbol{u}_i} \{\mathcal{V}_i\} = \min_{\boldsymbol{u}_i} \left\{ E_{\rho_k} \left\{ \mathcal{L}_i + \mathcal{V}_{i+1}^o \middle| \boldsymbol{I}_i \right\} \right\}, i = N-1, \dots, k,$$
$$\mathcal{V}_N^o = \boldsymbol{\mathcal{O}},$$

where the cost function at time i denoted $\mathcal{L}_i$ is defined as follows

$$\mathcal{L}_i = \left(\boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}\right)^T \left(\boldsymbol{V}_{i+1} - \boldsymbol{\Lambda}_{i+1}\right) \left(\boldsymbol{x}_{i+1} - \hat{\boldsymbol{x}}_{i+1|i}\right) + \\ + \left(\hat{\boldsymbol{s}}_{i+1|i} - \bar{\boldsymbol{s}}_{i+1}\right)^T \boldsymbol{Q}_{i+1} \left(\hat{\boldsymbol{s}}_{i+1|i} - \bar{\boldsymbol{s}}_{i+1}\right) + \boldsymbol{u}_i^T \boldsymbol{R}_i \boldsymbol{u}_i.$$

Bellman function

$$\mathcal{V}_i^o = \hat{\boldsymbol{s}}_i^T \boldsymbol{\Pi}_{N-i} \hat{\boldsymbol{s}}_i + \hat{\boldsymbol{s}}_i^T \boldsymbol{F}_{N-i} + \boldsymbol{F}_{N-i}^T \hat{\boldsymbol{s}}_i + h_{N-i}, i = N-1, \dots, k, \quad (5)$$

from the boundary condition follows that $\boldsymbol{\Pi}_0$, $\boldsymbol{F}_0$ and h_0 are zero valued

The dual control law

The dual control law

$$\boldsymbol{u}_k = -\Big[\, \boldsymbol{R}_k + \boldsymbol{B}^T(\hat{\boldsymbol{\theta}}_{k|k})\left(\boldsymbol{Q}_{k+1} + \boldsymbol{\Pi}_{N-k-1}\right)\boldsymbol{B}(\hat{\boldsymbol{\theta}}_{k|k}) + \boldsymbol{P}^{BB}_{k|k}\Big]^{-1} \times$$
$$\times \Big[\boldsymbol{B}^T(\hat{\boldsymbol{\theta}}_{k|k})\left(\boldsymbol{Q}_{k+1} + \boldsymbol{\Pi}_{N-k-1}\right)\boldsymbol{A}(\hat{\boldsymbol{\theta}}_{k|k})\hat{\boldsymbol{s}}_k + \boldsymbol{P}^{BA}_{k|k}\hat{\boldsymbol{s}}_k +$$
$$+ \boldsymbol{B}^T(\hat{\boldsymbol{\theta}}_{k|k})\left(\boldsymbol{Q}_{k+1} + \boldsymbol{\Pi}_{N-k-1}\right)\hat{\boldsymbol{w}}_k - \boldsymbol{B}^T(\hat{\boldsymbol{\theta}}_{k|k})\boldsymbol{Q}_{k+1}\bar{\boldsymbol{s}}_{k+1} +$$
$$+ \boldsymbol{B}^T(\hat{\boldsymbol{\theta}}_{k|k})\boldsymbol{F}_{N-k-1} + \boldsymbol{P}^{B\Theta}_{k|k}\Big].$$

Properties of the dual control law

- The control law is derived using the Bellman optimisation recursion.
- The dual properties manifested through $\boldsymbol{P}^{AA}_{i|k}$, $\boldsymbol{P}^{BA}_{i|k}$, $\boldsymbol{P}^{BB}_{i|k}$, $\boldsymbol{P}^{A\Theta}_{i|k}$ and $\boldsymbol{P}^{B\Theta}_{i|k}$ which depend on $\boldsymbol{P}_{i|k} = \operatorname{cov}_{\rho_k}\left(\boldsymbol{x}_i \middle| \boldsymbol{I}_k\right)$ for $i = N-1, \ldots, k$.
- Only first two moments of pdf's $p(\boldsymbol{x}_i | \boldsymbol{y}_0^k)$ are necessary.

Numerical example

Considered system

$$s_{k+1} = \begin{pmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{pmatrix} s_k + \begin{pmatrix} 0 \\ \theta_{3k} \end{pmatrix} u_k + \boldsymbol{w}_k$$
$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k$$
$$y_k = (0,\ 1)\boldsymbol{s}_k + v_k$$

- Initial state and the parameters
  - $\boldsymbol{s}_0 = (1,\ -0.5)^T$
  - $\boldsymbol{\theta}_0 = (-2.0427,\ 0.3427,\ 1)^T$
- Noise pdf's
  - $p(\boldsymbol{w}_k) = \mathcal{N}\left((0,\ 0)^T, 0.00012\mathbf{I}_2\right)$
  - $p(v_k) = \mathcal{N}\,(0,\ 0.001)$
- Prior pdf for EKF
  - $p(\boldsymbol{x}_0) = \mathcal{N}\left((1, -0.5, -2.0427, 0.3427, 1)^T, 0.2\mathbf{I}_5\right)$

Criteria parameters

Criterion of the original optimisation problem

$$J = E\left\{\sum_{k=0}^{N-1}(s_{k+1,2}-5)^2 + 0.001\cdot u_k^2\right\},$$

Modified criterion for dual control derivation

$$J_k = E_{\rho_k}\left\{\sum_{i=k}^{N-1}(s_{i+1,2}-5)^2 + 0.001\cdot u_i^2 - \boldsymbol{\nu}_{i+1}^T\boldsymbol{\Lambda}_{i+1}\boldsymbol{\nu}_{i+1}\middle|\boldsymbol{I}_i\right\}$$

$$\boldsymbol{\Lambda}_{i+1} = \begin{pmatrix} 0.5 & \boldsymbol{\Lambda}_{i+1}^{s,\theta} \\ \boldsymbol{\Lambda}_{i+1}^{s,\theta\,T} & \boldsymbol{\mathcal{O}} \end{pmatrix} \qquad \boldsymbol{\Lambda}_{i+1}^{s,\theta} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

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Comparison to other controllers

Control quality comparison using the quality measures $\hat{\mathcal{M}}$ and $\hat{\mathbb{C}}$

	$\hat{\mathcal{M}}$	$\hat{\mathbb{C}}$
CE	1.477650	8.245854
PCE	0.902443	1.193457
MSPEDC	0.882633	1.138688

- measure of meeting the control objective

$$\hat{\mathcal{M}} = \frac{1}{m}\left\{\sum_{j=1}^{m}\left(\frac{1}{N}\sum_{i=0}^{N-1}\left(s_{i+1,2}-5\right)^2\right)\right\}$$

- average cost of realising the system trajectory

$$\hat{\mathbb{C}} = \frac{1}{m}\left\{\sum_{j=1}^{m}\left(\sum_{i=0}^{N-1}\left(s_{i+1,2}-5\right)^2 + 0.001\cdot u_i^2\right)\right\}$$

Concluding remarks

Resume

- ➢ the new dual adaptive controller with multistage control horizon was introduced
- ➢ some aspects of the criterion and control law were discussed

Features of the new dual controller

- ✔ clear criterion interpretation
  - ➪ modified criterion incorporates both aspects of dual control
  - ➪ makes it possible to individually tune influence of parameter uncertainty on control
- ✔ closed form solution available
- ✔ higher control quality compared to CE and PCE controllers
- ✔ computationally moderate
- ✔ EKF if sufficient for the estimation of unknown state and parameters
- ✔ quite robust with respect to choice of weighting matrix $\boldsymbol{\Lambda}_{i+1}$