Optimal active decision making for control

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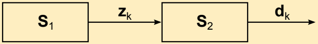
Presentation Overview

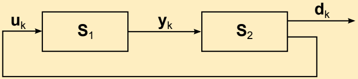
Introduction

Passive and active change detection or control

Passive Data $\mathbf{z}_k$ is passively used for generating decision $\mathbf{d}_k$



Active Decision $\mathbf{d}_k$ is based on input-output data $[\mathbf{u}_k, \mathbf{y}_k]$ and input $\mathbf{u}_k$ should improve quality of decisions or control system

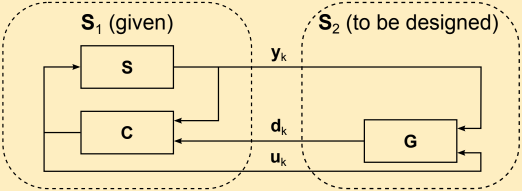


Introduction – cont'd

General formulation of active change detection and control

- Stems from stochastic optimal control formulation
- Assumes Closed loop information processing strategy
- Includes several design problems as special cases

Special case – Optimal active decision making for control



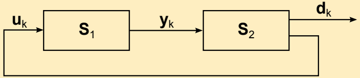
Introduction – cont'd

Goals

1. Outline the general formulation of the active change detection and control problem
2. Introduce the special case: Optimal active decision making for control
3. Present a suboptimal active generator

General formulation

Description of system $\mathbf{S}_1$ for time steps $k \in \mathcal{T} = \{0, \ldots, F\}$



$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{u}_k, \mathbf{w}_k\right)$$
$$\boldsymbol{\mu}_{k+1} = \mathbf{g}_k\left(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{u}_k, \mathbf{e}_k\right)$$
$$\mathbf{y}_k = \mathbf{h}_k\left(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{v}_k\right)$$

$\mathbf{f}_k$, $\mathbf{g}_k$, $\mathbf{h}_k$ – known vector functions
$\bar{\mathbf{x}}_k = \left[\mathbf{x}_k^T, \boldsymbol{\mu}_k^T\right]^T$ – system state, $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\boldsymbol{\mu}_k \in \mathcal{M} \subseteq \mathbb{R}^{n_\mu}$
$\mathbf{u}_k \in \mathcal{U}_k \subseteq \mathbb{R}^{n_u}$ – input, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ – output
$\mathbf{w}_k$, $\mathbf{e}_k$ – state noises with known pdf's $p(\mathbf{w}_k)$ and $p(\mathbf{e}_k)$
$\mathbf{v}_k$ – output noise with known pdf $p(\mathbf{v}_k)$
$\bar{\mathbf{x}}_0$ – initial condition with known pdf $p(\bar{\mathbf{x}}_0) = p(\mathbf{x}_0)p(\boldsymbol{\mu}_0)$

General formulation – cont'd

Description of system $\mathbf{S}_2$

$$\begin{bmatrix}\mathbf{d}_k \\ \mathbf{u}_k\end{bmatrix} = \boldsymbol{\rho}_k\left(\mathbf{I}_0^k\right)$$

$\boldsymbol{\rho}_k$ – a function to be designed
$\mathbf{I}_0^k = [{\mathbf{y}_0^k}^T, {\mathbf{u}_0^{k-1}}^T, {\mathbf{d}_0^{k-1}}^T]^T$ – an information vector
$\mathbf{d}_k$ – a decision (i.e. a point estimate of $\boldsymbol{\mu}_k$)

Criterion

$$J\left(\boldsymbol{\rho}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} \overbrace{\alpha_k L_k^{\mathrm{d}}\left(\boldsymbol{\mu}_k, \mathbf{d}_k\right) + (1-\alpha_k) L_k^{\mathrm{c}}\left(\mathbf{x}_k, \mathbf{u}_k\right)}^{L_k(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{u}_k, \mathbf{d}_k)}\right\}$$

General solution (Closed loop information processing strategy)

Backward recursive equation for time steps $k = F, F-1, \ldots, 0$

$$V_k^*\left(\mathbf{I}_0^k\right) = \min_{\substack{\mathbf{d}_k \in \mathcal{M} \\ \mathbf{u}_k \in \mathcal{U}_k}} \mathrm{E}\left\{L_k\left(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{u}_k, \mathbf{d}_k\right) + V_{k+1}^*\left(\mathbf{I}_0^{k+1}\right) \,\middle|\, \mathbf{I}_0^k, \mathbf{u}_k, \mathbf{d}_k\right\}$$

$V_{F+1}^* = 0$ – initial condition, $J(\rho_0^{F*}) = \mathrm{E}\left\{V_0^*\left(\mathbf{y}_0\right)\right\}$ – optimal value

Optimal system $\mathbf{S}_2$

$$\begin{bmatrix} \mathbf{d}_k^* \\ \mathbf{u}_k^* \end{bmatrix} = \arg \min_{\substack{\mathbf{d}_k \in \mathcal{M} \\ \mathbf{u}_k \in \mathcal{U}_k}} \mathrm{E}\left\{L_k\left(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{u}_k, \mathbf{d}_k\right) + V_{k+1}^*\left(\mathbf{I}_0^{k+1}\right) \,\middle|\, \mathbf{I}_0^k, \mathbf{u}_k, \mathbf{d}_k\right\}$$

Note: Pdf's $p\left(\bar{\mathbf{x}}_k | \mathbf{I}_0^k, \mathbf{u}_k, \mathbf{d}_k\right)$ and $p\left(\mathbf{y}_{k+1} | \mathbf{I}_0^k, \mathbf{u}_k, \mathbf{d}_k\right)$ are needed.

Optimal active decision making for control

From general formulation to optimal active decision making for control

- System S_2 consists of a given controller $\gamma_k(\mathbf{I}_0^k, \mathbf{d}_k)$

$$\begin{bmatrix} \mathbf{d}_k \\ \mathbf{u}_k \end{bmatrix} = \boldsymbol{\rho}_k\left(\mathbf{I}_0^k\right) = \begin{bmatrix} \boldsymbol{\sigma}_k\left(\mathbf{I}_0^k\right) \\ \gamma_k\left(\mathbf{I}_0^k, \boldsymbol{\sigma}_k\left(\mathbf{I}_0^k\right)\right) \end{bmatrix}$$

- Only the control aim is considered $\Rightarrow \alpha_k = 0$ (i.e. $L_k\left(\mathbf{x}_k, \boldsymbol{\mu}_k, \mathbf{u}_k, \mathbf{d}_k\right) = L_k^{\mathrm{c}}\left(\mathbf{x}_k, \mathbf{u}_k\right)$) and the criterion is

$$J\left(\boldsymbol{\sigma}_0^F\right) = \mathrm{E}\left\{\sum_{k=0}^{F} L_k^{\mathrm{c}}\left(\mathbf{x}_k, \mathbf{u}_k\right)\right\}$$

Optimal active decision making for control – cont'd

Backward recursive equation

$$V_k^*\left(\mathbf{I}_0^k\right) = \min_{\mathbf{d}_k \in \mathcal{M}} \mathrm{E}\left\{ L_k^c\left(\mathbf{x}_k, \gamma_k(\mathbf{I}_0^k, \mathbf{d}_k)\right) + V_{k+1}^*\left(\mathbf{I}_0^{k+1}\right) \,\Big|\, \mathbf{I}_0^k, \mathbf{u}_k, \mathbf{d}_k \right\}$$

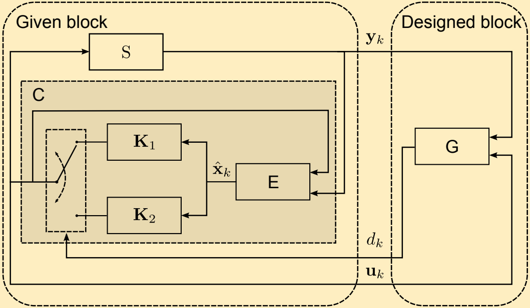
Optimal active generator for given controller

$$\mathbf{d}_k^* = \arg\min_{\mathbf{d}_k \in \mathcal{M}} \mathrm{E}\left\{ L_k^c\left(\mathbf{x}_k, \gamma_k(\mathbf{I}_0^k, \mathbf{d}_k)\right) + V_{k+1}^*\left(\mathbf{I}_0^{k+1}\right) \,\Big|\, \mathbf{I}_0^k, \mathbf{u}_k, \mathbf{d}_k \right\}$$

Note: The minimization over $\mathbf{d}_k$ is performed subject to the system and the given controller.

Given multimodel controller

A block diagram of active decision making for control



Given multimodel controller – cont'd

Description of system $\mathbf{S}_1$

$$\begin{aligned} \mathbf{x}_{k+1} =& \mathbf{A}_{\mu_k}\mathbf{x}_k + \mathbf{B}_{\mu_k}\mathbf{u}_k + \mathbf{G}_{\mu_k}\mathbf{w}_k \\ \mathbf{y}_k =& \mathbf{C}_{\mu_k}\mathbf{x}_k + \mathbf{H}_{\mu_k}\mathbf{v}_k \end{aligned}$$

$\mu_k \in \mathcal{M} = \{1, 2, \ldots, N\}$ – a model index
$P(\mu_{k+1} = j | \mu_k = i) = \pi_{i,j}$ – transition probabilities
$\mathbf{w}_k$, $\mathbf{v}_k$ – noises with Gaussian distribution $\mathcal{N}\{\mathbf{0}, \mathbf{I}\}$
$\mathbf{x}_0$ – initial state with Gaussian distribution $\mathcal{N}\{\hat{\mathbf{x}}_0', \mathbf{P}_0'\}$
μ_0 – initial model with probabilities $P(\mu_0)$

Given multimodel controller – cont'd

Additional assumptions

- Given controller

$$\mathbf{u}_k = \gamma_k \left(\mathbf{I}_0^k, d_k \right) = \mathbf{K}_{d_k} \hat{\mathbf{x}}_k$$

$\mathbf{K}_{d_k}$ – controller gain, $\hat{\mathbf{x}}_k = \mathrm{E}\{\mathbf{x}_k | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}\}$

- Quadratic cost function

$$L_k^{\mathrm{c}} (\mathbf{x}_k, \mathbf{u}_k) = [\mathbf{x}_k - \mathbf{r}_k]^T \mathbf{Q}_k [\mathbf{x}_k - \mathbf{r}_k] + \mathbf{u}_k^T \mathbf{R}_k \mathbf{u}_k$$

$\mathbf{r}_k$ – known function of time

- Optimization horizon $F_o = 2$ means that $V_{k+2}^*(\mathbf{I}_0^{k+2})$ is replaced by the zero function – rolling horizon technique

Approximation based on rolling horizon

Time step $k+1$

- Approximate cost-to-go function

$$\tilde{V}_{k+1}\left(\mathbf{I}_0^{k+1}\right) = [\hat{\mathbf{x}}_{k+1} - \mathbf{r}_{k+1}]^T \mathbf{Q}_{k+1} [\hat{\mathbf{x}}_{k+1} - \mathbf{r}_{k+1}] + \mathrm{tr}\left(\mathbf{Q}_{k+1}\mathbf{P}_{k+1}\right) + \min_{d_{k+1}} \left\{ \hat{\mathbf{x}}_{k+1}^T \mathbf{K}_{d_{k+1}}^T \mathbf{R}_{k+1} \mathbf{K}_{d_{k+1}} \hat{\mathbf{x}}_{k+1} \right\}$$

- Decision

$$\tilde{\mathbf{d}}_{k+1} = \arg\min_{d_{k+1}} \left\{ \hat{\mathbf{x}}_{k+1}^T \mathbf{K}_{d_{k+1}}^T \mathbf{R}_{k+1} \mathbf{K}_{d_{k+1}} \hat{\mathbf{x}}_{k+1} \right\}$$

Note: Mean value $\hat{\mathbf{x}}_k = \mathrm{E}\{\mathbf{x}_k | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}\}$ and covariance matrix $\mathbf{P}_k = \mathrm{cov}\{\mathbf{x}_k | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}\}$ can be obtained from estimation algorithm.

Approximation based on rolling horizon – cont'd

Time step k

- Approximate cost-to-go function

$$\tilde{V}_k\left(\mathbf{I}_0^k\right) = [\hat{\mathbf{x}}_k - \mathbf{r}_k]^T \mathbf{Q}_k [\hat{\mathbf{x}}_k - \mathbf{r}_k] + \mathrm{tr}\left(\mathbf{Q}_k \mathbf{P}_k\right) + \min_{d_k}\left\{\hat{\mathbf{x}}_k^T \mathbf{K}_{d_k}^T \mathbf{R}_k \mathbf{K}_{d_k} \hat{\mathbf{x}}_k + \underbrace{\mathrm{E}\left\{\tilde{V}_{k+1}\left(\mathbf{I}_0^{k+1}\right) | \mathbf{I}_0^k, \mathbf{u}_k, d_k\right\}}_{\Omega_{d_k}(\mathbf{I}_0^k, \mathbf{u}_k, d_k)}\right\}.$$

- Decision

$$\tilde{\mathbf{d}}_k = \arg\min_{d_k}\left\{\hat{\mathbf{x}}_k^T \mathbf{K}_{d_k}^T \mathbf{R}_k \mathbf{K}_{d_k} \hat{\mathbf{x}}_k + \Omega_{d_k}(\mathbf{I}_0^k, \mathbf{u}_k, d_k)\right\}$$

Approximation based on rolling horizon – cont'd

Time step k

- Expected cost-to-go $\Omega_{d_k}(\mathbf{I}_0^k, \mathbf{u}_k, d_k)$

$$\begin{aligned}\Omega_{d_k}(\mathbf{I}_0^k, \mathbf{u}_k, d_k) = \\ & [\hat{\mathbf{x}}'_{k+1} - \mathbf{r}_{k+1}]^T \mathbf{Q}_{k+1} [\hat{\mathbf{x}}'_{k+1} - \mathbf{r}_{k+1}] + \mathrm{tr}\left(\mathbf{Q}_{k+1} \mathbf{P}'_{k+1}\right) \\ & + \mathrm{E}\left\{ \min_{d_{k+1}} \left\{ \hat{\mathbf{x}}_{k+1}^T \mathbf{K}_{d_{k+1}}^T \mathbf{R}_{k+1} \mathbf{K}_{d_{k+1}} \hat{\mathbf{x}}_{k+1} \right\} \mid \mathbf{I}_0^k, \mathbf{u}_k, d_k \right\}\end{aligned}$$

- The expectation $\mathrm{E}\{\min_{d_{k+1}}\{\cdot\}|\mathbf{I}_0^k, \mathbf{u}_k, d_k\}$ is computed numerically for each decision $d_k \in \mathcal{M}$

Numerical example

Description of the system $\mathbf{S}_1$ for the finite horizon $F = 19$

- Parameters of two models

$$\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{G}_1 = \mathbf{G}_2 = 0.01\mathbf{E}_2,$$
$$\mathbf{C}_1 = \mathbf{C}_2 = [1\ 0], \quad \mathbf{H}_1 = \mathbf{H}_2 = 0.01,$$
$$\mathbf{B}_1 = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

- Transition probabilities $\pi_{1,1} = \pi_{2,2} = 0.96,\ \pi_{1,2} = \pi_{2,1} = 0.04$
- Initial conditions $P\left(\mu_0 = i\right) = 0.5,\ i = 1, 2,\ \hat{\mathbf{x}}'_0 = [1,\ 0]^T$ and $\mathbf{P}'_0 = 10^{-4}\mathbf{E}_2$
- Reference signal $r_k = 0$, matrices $\mathbf{Q}_k = \mathbf{E}_2$, $\mathbf{R}_k = 0.1$ for all k

Numerical example – cont'd

Active and passive generator comparison

- Passive generator (PG1)

$$d_k^{\mathrm{PG1}} = \arg\min_{\mu_k} P(\mu_k | y_0^k, u_0^{k-1})$$
$$u_k^{\mathrm{PG1}} = \mathbf{K}_{d_k^{\mathrm{PG1}}} \hat{\mathbf{x}}_k$$

- Active generator (AG)

$$d_k^{\mathrm{AG}} = \arg\min_{d_k} \left\{ \hat{\mathbf{x}}_k^T \mathbf{K}_{d_k}^T \mathbf{R}_k \mathbf{K}_{d_k} \hat{\mathbf{x}}_k + \Omega_{d_k}(\mathbf{I}_0^k, \mathbf{u}_k, d_k) \right\}$$
$$u_k^{\mathrm{AG}} = \mathbf{K}_{d_k^{\mathrm{AG}}} \hat{\mathbf{x}}_k$$

Active and passive generator comparison – cont'd

Results of M = 10000 MC simulations

Geberator	$\hat{J}$	$\mathrm{var}\{\hat{J}\}$	$\mathrm{var}\{L\}$	N_{wd}
PG1	7.42	0.031	317	2.43
AG	4.23	0.002	10	10.31

$$J = \mathrm{E}\left\{ \overbrace{\sum_{k=0}^{F} L_k^{\mathrm{c}}\left(\mathbf{x}_k, \mathbf{u}_k\right)}^{L} \right\} \qquad \hat{J} = \frac{1}{M}\sum_{i=1}^{M} L^i$$

$$\mathrm{var}\left\{\hat{J}\right\} = \mathrm{bootstrap}\left\{L^i\right\} \qquad \mathrm{var}\{L\} = \frac{1}{M-1}\sum_{i=1}^{M}\left(L^i - \hat{J}\right)^2$$

N_{wd} – number of wrong decisions (i.e. $d_k \neq \mu_k$)

Numerical example – cont'd

Alternative passive generator

- Passive generator (PG2)

$$d_k^{\mathrm{PG2}} = \arg\min_{\mu_k} P(\mu_k | y_0^k, u_0^{k-1})$$

$$u_k^{\mathrm{PG2}} = \sum_{\mu_k} \mathbf{K}_{\mu_k} \hat{\mathbf{x}}_{\mu_k} P(\mu_k | y_0^k, u_0^{k-1})$$

where $\hat{\mathbf{x}}_{\mu_k} = \mathrm{E}\{\mathbf{x}_k | \mathbf{y}_0^k, \mathbf{u}_0^{k-1}, \mu_k\}$

Numerical example – cont'd

Results of M = 10000 MC simulations

Geberator	$\hat{J}$	$\text{var}\{\hat{J}\}$	$\text{var}\{L\}$	N_{wd}
PG1	7.42	0.031	317	2.43
PG2	4.50	0.013	59	2.32
AG	4.23	0.002	10	10.31

$$J = \text{E}\left\{ \overbrace{\sum_{k=0}^{F} L_k^{\text{c}}\left(\mathbf{x}_k, \mathbf{u}_k\right)}^{L} \right\} \qquad \hat{J} = \frac{1}{M} \sum_{i=1}^{M} L^i$$

$$\text{var}\left\{\hat{J}\right\} = \text{bootstrap}\left\{L^i\right\} \qquad \text{var}\{L\} = \frac{1}{M-1} \sum_{i=1}^{M} \left(L^i - \hat{J}\right)^2$$

N_{wd} – number of wrong decisions (i.e. $d_k \neq \mu_k$)

Conclusion remarks

Conclusion

- The general formulation of active change detection and control
- Special case: Optimal active decision making for control
- The suboptimal active generator (rolling horizon technique)
- The numerical example showing the advantage of active generator

Further work

- Focus on the case with a given detector
- Approximative techniques