# Implicational Interpretation - Continuity Issue

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# Outline

# Goals and activities

## 1. Approximate reasoning and fuzzy approximation

1. Applications of fuzzy logic in broader sense: fuzzy quantifiers, deduction, syllogisms, fuzzy inference systems (Novák, Perfilieva, Dvořák, Štěpnička, Pavliska, Daňková)
2. Higher order fuzzy logics: interpretations and properties of a fragment of logic in models based on $\Omega$-sets (Močkoř)
3. Fuzzy relation equations, algorithm of recognition of linearly dependent and independent vectors in semi-linear spaces (Perfilieva, Štěpnička, Kupka)
4. Use of fuzzy transforms: numerical methods for differential equations and other applications (Perfilieva, Štěpnička)

# Goals and activities

## 2. Combination of stochastic and fuzzy models

1. Mining linguistic associations from data (Novák, Dvořák, Perfilieva, Kupka)
2. Soft computing methods in image processing (Perfilieva, Pavliska, Vajgl, Daňková)
3. Soft computing methods for time series analysis (Perfilieva, Novák, Dvořák, Pavliska, Štěpnička)

## 3. Fuzzy modeling of complex processes

1. Development of the LFLC software package and co-operating applications (Pavliska, Dvořák, Huňka)

# This presentation

## Implicational Interpretation - Continuity Issue

### Goal

1. Approximate reasoning and fuzzy approximation

### Activity

Applications of fuzzy logic in broader sense: fuzzy quantifiers, deduction, syllogisms, fuzzy inference systems

# Outline

# Fuzzy rule(s)

$$\mathcal{R} := \text{IF } x \text{ is } \mathcal{A} \text{ THEN } y \text{ is } \mathcal{B} \tag{1}$$

... usually, we have a finite set of them

$$\mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_n\} \tag{2}$$

and we talk about the so called *linguistic description* (or a *fuzzy rule base*)

# Interpretation

Usually so called Relational Interpretation is considered:

1. Sets $X$ and $Y$ are inupt/output universes, respectively
2. Linguistic expressions $\mathcal{A}, \mathcal{B}$ are interpreted by fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, respectively
3. Fuzzy rules $\mathcal{R}_i$ are interpreted by fuzzy relations $R_i \in \mathcal{F}(X \times Y)$ involving fuzzy sets which interpret the linguistic expressions in respective fuzzy rules
4. Linguistic description $\mathcal{R}$ is interpreted by a fuzzy relation $R \in \mathcal{F}(X \times Y)$ involving fuzzy relations $R_i$

# Inference

In case of the *relational interpretation* the inference is usually modelled as an image of a fuzzy set under a fuzzy relation:

$$B' = A' \circ R$$

$A' \in \mathcal{F}(X)$ - fuzzy input
$B' \in \mathcal{F}(Y)$ - fuzzy output, deduced by the inference $\circ$ with help of the relational interpretation $R$

Each appropriate inference has the property that in case of a crisp input $x' \in X$ the output $B'$ is deduced based ONLY on the fuzzy relation $R$:

$$B'(y) = R(x', y)$$

# Main Interpretations

Two main interpretations

**DNF (Mamdani-Assilian interpretation)**

$$\check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y))$$

**CNF (Implicational interpretation)**

$$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \rightarrow B_i(y))$$

# CNF & DNF

- There is no significant difference between CNF and DNF interpretations (from the practical point of view)
- DNF is widely applied while CNF is ignored
- To each continuous and piecewise monotonous function $f : X \longrightarrow Y$ there exists $\hat{R}$ and an appropriate defuzzification $\mathrm{DEF}$ such that $\mathrm{DEF}_Y(\hat{R}(x, \cdot)) = f(x)$ for each $x \in X$

... strange reasons and arguments for the ignorance

# Consistency

Consistent linguistic description:

- contains no conflict in rules
- ... no rules with the same (or similar) antecedents and contradictory consequents

**Inconsistent linguistic description**

IF *obstacle* is *left* OR *front* THEN *bypass* is *right*
IF *obstacle* is *right* OR *front* THEN *bypass* is *left*.

# Consistency

D. Dubois, H. Prade: CNF interpretation of inconsistent linguistic description **lowers** the largest membership degree.

Consistency of a linguistic description defined via so called coherence.

## Coherence

$\hat{R} \in \mathcal{F}(X \times Y)$ - interpretation of linguistic description (2).
$\hat{R}$ is *coherent* if to each $x \in X$ there exists $y \in Y$ such that

$$\hat{R}(x, y) = 1$$

# Coherence

Coherence: $\mathrm{Core}(\hat{R}(x,\cdot))$ is non-empty!

Fix $x$, the set of $y \in Y$ such that $A_i(x) \to B_i(y) = 1$ is the set $y$ such that

$$A_i(x) \leq B_i(y),$$

i.e., the set of those outputs which fully satisfy the $i$-th rule. The higher $A_i(x)$, the narrower is the set of such $y$

For a given $x \in X$, $\mathrm{Core}(\hat{R}(x,\cdot)) \neq \emptyset$ is

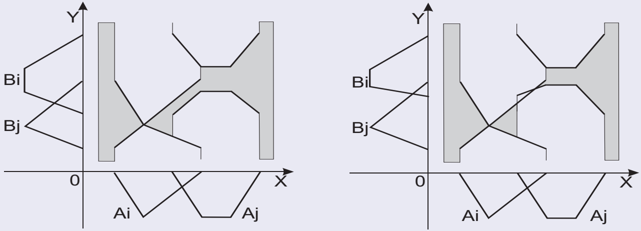
$$\bigcap_{i=1}^{n} \{y \mid A_i(x) \to B_i(y) = 1\} \neq \emptyset.$$

There has to be $y \in Y$ fully satisfying all rules.

# Coherence (D. Dubois, H. Prade)

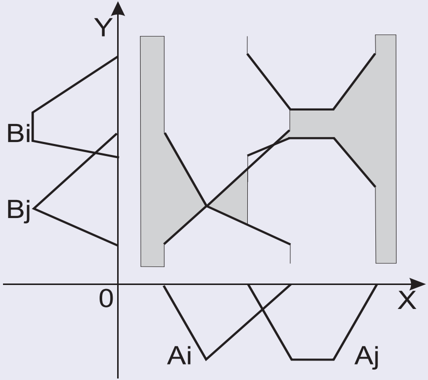
For all $x' \in X$ there exists $y \in Y$ such that $R(x', y) = 1$

## Figure from D. Dubois, H. Prade and L. Ughetto - IEEE '97



# Coherence = consistency

## Incoherent rule base



# Coherence - remarks

**Note:**

1. Coherence of $\hat{R}$ can be easily checked or even fulfilled in advance when constructing a linguistic description (see D. Dubois, H. Prade, L. Ughetto, D. Coufal)
2. The construction of the coherence explains, why the $\mathrm{MOM}$ defuzzification is applied when dealing with CNF (averages only nodes with maximal - equal to 1 - values)
3. For $\check{R}$, the coherence as defined above is inappropriate
4. D. Coufal proposed a coherence index for $\check{R}$ based on the convexity, but it is not that easy to check

# Outline

# Basic definitions and notations

$$\mathcal{L} = \langle [0,1], \vee, \wedge, *, \rightarrow, 0, 1 \rangle \tag{3}$$

$X$ be a compact subset of real numbers $\mathbb{R}^m$

Let $A \in \mathcal{F}(X)$ and $\alpha \in [0,1]$. Then

$$\begin{aligned} \mathrm{height}(A) &= \sup\{A(x) \mid x \in X\}, \\ \mathrm{Core}(A) &= \{x \mid A(x) = 1\}, \\ \mathrm{Supp}(A) &= \{x \mid A(x) > 0\}, \\ \mathrm{Ceil}(A) &= \{x \mid A(x) = \mathrm{height}(A)\}, \\ [A]_\alpha &= \{x \mid A(x) \geq \alpha\}. \end{aligned}$$

# General setting

## Convexity

$A \in \mathcal{F}(X)$ is convex if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq A(x_1) \wedge A(x_2), \quad \lambda \in [0,1],\ x_1, x_2 \in X$$

## Strict convexity

$A \underset{\sim}{\subset} U$ is strictly convex if

$$A(\lambda x_1 + (1-\lambda)x_2) > A(x_1) \wedge A(x_2), \quad \lambda \in (0,1),\ x_1, x_2 \in Supp(A),$$

and $x_1, x_2 \notin Core(A)$

... removes constant parts excepting the core and the support

# Defuzzification

**(i)** Let $\mathrm{Ceil}(A) \neq \emptyset$. Then the *mean of maxima* of $A$ is a function

$$\mathrm{MOM}_X(A) = \begin{cases} \frac{\sum_{x \in \mathrm{Ceil}(A)} x}{|\mathrm{Ceil}(A)|} & \text{if } \int_{\mathrm{Ceil}(A)} 1 \; dx = 0, \\ \frac{\int_{\mathrm{Ceil}(A)} x \; dx}{\int_{\mathrm{Ceil}(A)} 1 \; dx} & \text{otherwise.} \end{cases}$$

**(ii)** Let $\int_X A(x)\; dx > 0$. Then the *center of gravity* of $A$ is a function

$$\mathrm{COG}_X(A) = \frac{\int_X x \cdot A(x)\; dx}{\int_X A(x)\; dx}.$$

# General setting and notations

For all further consideration,

**we assume:**

- $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}$
- $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ are compact normed spaced
- i.e. $X$ is a compact set on $\mathbb{R}^m$ and $Y$ is a closed interval
- $\partial C$ denotes the boundary of an arbitrary set $C$ and $\overline{C}$ denotes the closure of the set $C$, in a given normed spaces

# General setting and notations

Let $\hat{R}$ be coherent and let $\hat{R}_i \in \mathcal{F}(X \times Y)$ be given by

$$\hat{R}_i(x, y) = A_i(x) \to B_i(y).$$

Then functions $\mathrm{I\hat{R}}_i, \mathrm{S\hat{R}}_i, \mathrm{I\hat{R}}, \mathrm{S\hat{R}}, \mathrm{M\hat{R}} : X \to Y$ are defined as follows

$$\mathrm{I\hat{R}}_i(x) = \inf \mathrm{Core}(\hat{R}_i(x, \cdot)), \tag{4}$$

$$\mathrm{S\hat{R}}_i(x) = \sup \mathrm{Core}(\hat{R}_i(x, \cdot)), \tag{5}$$

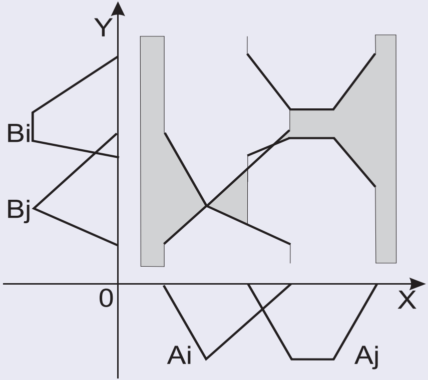
$$\mathrm{I\hat{R}}(x) = \inf \mathrm{Core}(\hat{R}(x, \cdot)), \tag{6}$$

$$\mathrm{S\hat{R}}(x) = \sup \mathrm{Core}(\hat{R}(x, \cdot)), \tag{7}$$

$$\mathrm{M\hat{R}}(x) = \frac{\mathrm{I\hat{R}}(x) + \mathrm{S\hat{R}}(x)}{2}. \tag{8}$$

# Coherence = consistency

## Functions defined above



# Outline

# Auxiliary results

## Lemma 1

$B_i \in \mathcal{F}(Y)$, $i = 1, \ldots, n$ be normal and convex.

Then for all $x \in X$ and arbitrary $\alpha \in [0, 1]$, either $[\hat{R}(x, \cdot)]_\alpha = \emptyset$ or it is a closed interval.

## Corollary 1

Assumptions of Lemma 1 hold.

Then

$$\mathrm{MOM}_Y(\hat{R}(x, \cdot)) = \mathrm{M\hat{R}}(x), \quad x \in X. \tag{9}$$

# Auxiliary results

- focus restricted to coherent f. relations based on convex and normal consequent fuzzy sets $\Rightarrow$ defuzzified output is only the arithmetic mean of inf and sup of the $\mathrm{Core}$

**Fact**

Assumptions of Lemma 1 hold and $\mathrm{I\hat{R}}, \mathrm{S\hat{R}}$ are continuous.

Then the defuzzified output $\mathrm{MOM}_Y(\hat{R}(x, \cdot))$ is a continuous function too.

# Restrictions

Let us restrict our focus to:

- continuous normal consequent fuzzy sets $B_i$ that are strictly convex (membership functions are continuous and strictly monotone to the left and to the right of $\mathrm{Core}(B_i)$).

It means that in the rest of the presentation, we will consider:

$$B_i(y) = \begin{cases} B_i^L(y) & \text{if } y \in (\inf(\mathrm{Supp}(B_i)), b_i^L], \\ 1 & \text{if } y \in (b_i^L, b_i^R), \\ B_i^R(y) & \text{if } y \in [b_i^R, \sup(\mathrm{Supp}(B_i))), \end{cases}$$

# Main results

**Lemma 2**

Let $A_i \in \mathcal{F}(X)$, $i = 1, \ldots, n$ be continuous.
Let $B_i \in \mathcal{F}(Y)$ be continuous, normal and strictly convex.
Let $\hat{R}$ be coherent.

Then each function $\mathrm{I\hat{R}}_i$ and $\mathrm{S\hat{R}}_i$ is continuous for all

$$x \in X \setminus \partial(\mathrm{Supp}(A_i)).$$

Strict convexity plays an essential role in the proof.

# Main results

## Theorem 1

Let $A_i \in \mathcal{F}(X)$, $i = 1, \ldots, n$ be continuous. Let $B_i \in \mathcal{F}(Y)$ be continuous, normal and strictly convex. Let $\hat{R}$ be coherent. Then functions $\mathrm{I}\hat{\mathrm{R}}$ and $\mathrm{S}\hat{\mathrm{R}}$ are continuous for all

$$x \in X \setminus \bigcup_{i=1}^{n} \partial(\mathrm{Supp}(A_i)).$$

## Corollary 2

Assumptions of Theorem 1 hold. Let $\mathrm{Supp}(A_i) = X$ for all $i = 1, \ldots, n$.

Then the defuzzified output $\mathrm{MOM}_Y(\hat{R}(x, \cdot))$ is a continuous function on $X$.

# Alternative approach

Previous results - removing points of possible discontinuities

**Remark**

The combination of the Ruspinin condition

$$\sum_{i=1}^{n} A_i(x) = 1, \quad \forall x \in X$$

and the normality of antecedents $A_i$ is a common requirement but also of a high theoretical and practical importance (interpolation).

Alternative approach aiming at the output axis $Y$ should be investigated.

# Alternative approach

**Lemma 3**

Assumptions of Lemma 2 hold ($A_i$ - continuous; $B_i$ - continuous, normal and strictly convex; $\hat{R}$ - coherent) and $\overline{\mathrm{Supp}(B_i)} = Y$.

Then functions $\mathrm{I}\hat{\mathrm{R}}_i$ and $\mathrm{S}\hat{\mathrm{R}}_i$ are continuous on $X$ for all $i = 1, \ldots, n$.

**Corollary 3**

With the assumptions of Lemma 3

the defuzzified output $\mathrm{MOM}_Y(\hat{R}(x, \cdot))$ is a continuous function.

# Criterion for continuity of $\mathrm{I}\hat{\mathrm{R}}_i$ and $\mathrm{S}\hat{\mathrm{R}}_i$

**Lemma 4**

Assumptions of Lemma 2 hold ($A_i$ - continuous; $B_i$ - continuous, normal and strictly convex; $\hat{R}$ - coherent)

Functions $\mathrm{I}\hat{\mathrm{R}}_i$ and $\mathrm{S}\hat{\mathrm{R}}_i$ are continuous **if and only if** at least one of the following conditions is fulfilled:

1. $\mathrm{Supp}(A_i) = X$,
2. $\overline{\mathrm{Supp}(B_i)} = Y$.

# Sufficient for continuity of $\mathrm{I}\hat{\mathrm{R}}$ and $\mathrm{S}\hat{\mathrm{R}}$

**Corollary 4**

Assumptions of Lemma 2 hold ($A_i$ - continuous; $B_i$ - continuous, normal and strictly convex; $\hat{R}$ - coherent)

If for all $i = 1, \ldots, n$ at least one of the following conditions is fulfilled:

1. $\mathrm{Supp}(A_i) = X$,
2. $\overline{\mathrm{Supp}(B_i)} = Y$

then the defuzzified output $\mathrm{MOM}_Y(\hat{R}(x, \cdot))$ is a continuous function.

# Further results

- **Lemma 4** specified the necessary and sufficient conditions for the continuity of functions $\mathrm{I}\hat{\mathrm{R}}_i$ and $\mathrm{S}\hat{\mathrm{R}}_i$.
- **Corollary 4** stated the sufficient ones for the continuity of functions $\mathrm{I}\hat{\mathrm{R}}$ and $\mathrm{S}\hat{\mathrm{R}}$.
- Specification of necessary and sufficient conditions for the continuity of $\mathrm{I}\hat{\mathrm{R}}$ and $\mathrm{S}\hat{\mathrm{R}}$ is more complicated. (Continuity of $\mathrm{I}\hat{\mathrm{R}}_i$ and $\mathrm{S}\hat{\mathrm{R}}_i$ is not needed.)

# Further results

**Theorem 2**

Assumptions of Lemma 2 hold ($A_i$ - continuous; $B_i$ - continuous, normal and strictly convex; $\hat{R}$ - coherent)

Functions $\mathrm{I}\hat{\mathrm{R}}$ and $\mathrm{S}\hat{\mathrm{R}}$ are continuous **if and only if** for all $i = 1, \ldots, n$ at least one of the following conditions is fulfilled:

... to be continued

# Theorem 2 ... continuation

## ... necessary and sufficient conditions

1. $\mathrm{Supp}(A_i) = X$,
2. $\overline{\mathrm{Supp}(B_i)} = Y$
3. for all $x \in \partial(\mathrm{Supp}(A_i))$ there exist $k, \ell \in \{1, \dots, n\}$ and there exists an $\varepsilon > 0$ open neighborhood of $x$ (denoted by $\mathcal{U}_\varepsilon(x)$) that

$$\mathrm{I\hat{R}}_k(x') \geq \mathrm{I\hat{R}}_i(x'), \quad x' \in \mathcal{U}_\varepsilon(x)$$

and

$$\mathrm{S\hat{R}}_\ell(x') \leq \mathrm{S\hat{R}}_i(x'), \quad x' \in \mathcal{U}_\varepsilon(x).$$

# Summary

## Results

- Theorem 2 specifies necessary and sufficient conditions for the continuity of $\mathrm{I}\hat{\mathrm{R}}$ and $\mathrm{S}\hat{\mathrm{R}}$
- Obviously, these are only sufficient for the continuity of $\mathrm{MOM}_Y(\hat{R}(x, \cdot))$
- However, we have a clear idea, how to ensure the continuity based on practical sufficient conditions (specified by corollaries)

# Outline

# Summary

## Coherence

- useful property imposing the consistency of a linguistic description
- helps us to solve the continuity issue after the MOM defuzzification
- output after defuzzification is independent on the chosen residuation $\rightarrow$
- cannot be as easily defined and checked for the DNF as it is possible for the CNF

# Summary

## DNF and CNF

- DNF and CNF keep the same practical properties
- CNF is for real applications ignored
- most of the arguments against CNF are not correct and have been controverted (FUZZ-IEEE'07)
- continuity of the defuzzified output was among often repeated arguments for advantage of DNF

But...

# Summary

## Continuity

- automatic continuity of DNF + $\mathrm{COG}$ might be dangerous (in case of inconsistent rules)
- due to the coherence, consistency can be easily handled for CNF
- in case of coherent $\hat{R}$, the continuity can be enforced
- strict convex fuzzy sets with unlimited supports played a crucial role

# Thanksgiving

**Thanksgiving**

Thank You for Your Attention!