Estimation of Models with Uniform Innovations and its Application on Traffic Data

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PRESENTATION LAYOUT

- Motivation, aims of the work
- LU model description
- Approximate estimation of LU model
- Illustrative examples
- Conclusions

MOTIVATION

- models with normal innovations
  - ⊕ simplicity
  - ⊖ unsuitable for bounded quantities
- unknown-but-bounded errors
  - ⊕ suitable for bounded quantities
  - ⊖ missing statistical tools
- model "XYZ"
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AIMS

probabilistic approach + bounded errors = LU model

Aims of the research

- to propose the probabilistic linear state-space model with bounded innovations
- to design algorithms for the estimation of the unknown quantities of this model
- to demonstrate the functionality of mentioned algorithms on the simulated and traffic data

LINEAR UNIFORM STATE-SPACE MODEL

$$\begin{array}{rcll} x_t & = & {}^cA_t x_{t-1} & + {}^cB_t u_t + {}^cF_t + {}^xe_t \\ y_t & = & {}^cC_t x_t & + {}^cD_t u_t + {}^cG_t + {}^ye_t, \end{array}$$

where

x_t, u_t, y_t - state, input, output
${}^cA_t$, ${}^cB_t$, e.t.c. - model matrices

$${}^cA_t = A_t + {}^eA,\ {}^cB_t = B_t + {}^eB, \text{ e.t.c.}$$

${}^xe_t$, ${}^ye_t$ - state and output innovations

The innovations have **uniform distributions**

$$f\left({}^xe_t\right) = \mathcal{U}\left(0, {}^xr\right), \quad f\left({}^ye_t\right) = \mathcal{U}\left(0, {}^yr\right)$$

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Collection of all estimated parameters:

$$\theta \equiv [\,{}^{e}A,\ {}^{e}B,\ {}^{e}F,\ {}^{e}C,\ {}^{e}D,\ {}^{e}G]'\,,$$

$$\Theta \equiv \left[\theta',\ {}^{x}r',\ {}^{y}r'\right]'$$

Assumptions:

- natural conditions of control hold
- x_0, ${}^{x}r$, ${}^{y}r$, Θ are mutually independent

LU MODEL - JOINT PDF

$$f(d^{1:T}, x^{0:T}, \Theta) = \mathrm{const}\ \chi(\mathcal{S})$$

$\mathcal{S} = \mathcal{S}0 \cap \mathcal{S}1 \cap \mathcal{S}2$:

$$\mathcal{S}0 = \left\{ \underline{x}_0 \le x_0 \le \overline{x}_0,\ \underline{\Theta} \le \Theta \le \overline{\Theta} \right\}$$

$$\mathcal{S}1 = \{ \begin{array}{l} -{}^x r \le x_t - {}^cA_t x_{t-1} - {}^cB_t u_t - {}^cF_t \le {}^x r \\ -{}^y r \le y_t - {}^cC_t x_t - {}^cD_t u_t - {}^cG_t \le {}^y r \end{array} \}$$

$$\mathcal{S}2 = \{ \underline{x} \le x_t \le \overline{x} \}$$

with $t = \{1, 2, \ldots, T\}$

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OFF-LINE ESTIMATION

- state estimation
- parameter estimation

APPROXIMATE PDF

Sliding window (memory length) - δ

$$f(d^{t-\delta:t}, x^{t-\delta:t}, \Theta) \approx \text{const}\ \chi(\tilde{\mathcal{S}}_t)$$

$\tilde{\mathcal{S}}_t = \tilde{\mathcal{S}}0_t \cap \tilde{\mathcal{S}}1_t \cap \tilde{\mathcal{S}}2_t$:

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with $\tau \in \{t-\delta, \dots, t\}$, $t \in \{\delta+1, \dots, T\}$

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ON-LINE ESTIMATION

- state estimation
- parameter estimation
- joint state and parameter estimation

ON-LINE JOINT ESTIMATION

- evaluation of the pdf

$$f(x^{t-\delta:t}, \Theta | d^{t-\delta:t}),\ t \in \{\delta+1, \ldots, T\},\ \delta > 1.$$

- MAP estimates - solved by linear programming (LP)

Find a vector X such that $J \equiv \mathcal{C}'X$

$$= \sum_{i=1}^{x^\ell} {}^x r_i + \sum_{j=1}^{y^\ell} {}^y r_j \rightarrow \min$$

while $\mathcal{A}X \leq \mathcal{B}$, $\underline{X} \leq X \leq \overline{X}$,

$\mathcal{C}' \equiv [\mathbf{0}'_{(X^\ell - x^\ell - y^\ell)}, \mathbf{1}'_{(x^\ell + y^\ell)}]$,
$\mathcal{A}$, $\mathcal{B}$ - from $\mathcal{S}1_t$
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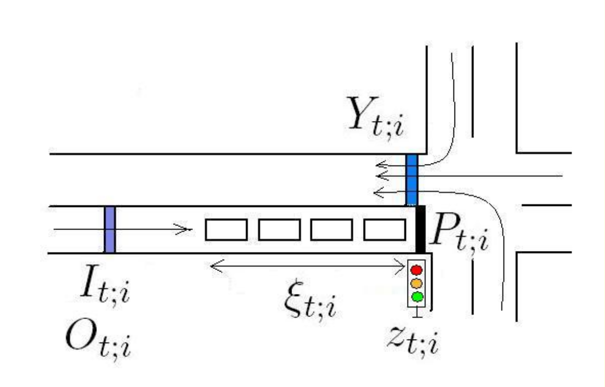
LINEARIZATION FOR LP

$$
\begin{aligned}
{}^{e}Ax_{t-1} &\approx {}^{e}A\hat{x}_{t-1} - {}^{e}\hat{A}\hat{x}_{t-1} + {}^{e}\hat{A}x_{t-1}, t \in t^{*}, \\
{}^{e}Cx_{t} &\approx {}^{e}C\hat{x}_{t} - {}^{e}\hat{C}\hat{x}_{t} + {}^{e}\hat{C}x_{t}, t \in t^{*},
\end{aligned}
$$

Model of **controlled intersection** - quantities:

- measured - intensity I_t and Y_t, occupancy O_t
- estimated - length of the car queue ξ_t, parameters κ, β, λ
- given - green time z_t, sat. flow S, turning rates α



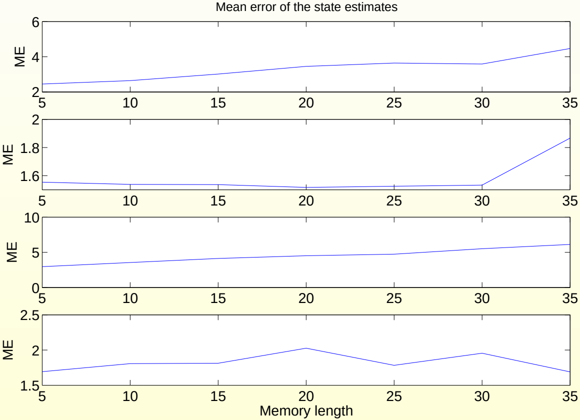


Figure: Mean errors of queue length estimates depending on memory length δ

Discussion of the results

ME of ξ estimates reach approximately 20% of the mean value ξ (50% for the 3rd arm)

Improving by:

- including I_t into the state vector
- estimation of turning rates α

Benefits of LU model

- promising alternative for the KF in the case of bounded data
- computationally feasible Matlab algorithms (MAP $\rightarrow$ LP)
- complying with hard prior bounds on model parameters and states
- reduction of the model ambiguity
- on-line estimation - update on the whole window δ
- estimation of the innovation ranges included

Possible exploitation of LU model

- off-line estimates of the innovations boundaries - initial setting of covariances for Kalman filtering
- application on the traffic data is promising
- starting point for estimation of a class of non-uniform distributions with restricted support

CONCLUSIONS - FUTURE PLANS

Improving of the estimates quality

General LU model:

- the method of selection of the inequalities for LP (informative data)
- the computation of the parameter estimates precision
- the approximation of the non-uniform pdf by the uniform one
- the use of the nonlinear programming

LU model of the intersection:

- the introduction of the model for the input intensity
- the on-line estimation of α

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Thank you for your attention!