

# FUZZY COALITIONAL STRUCTURES

(ALTERNATIVES)\*

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**Abstract.** The uncertainty of expectations and vagueness of the interests belong to natural components of cooperative situations, in general. Therefore, some kind of formalization of uncertainty and vagueness should be included in realistic models of cooperative behaviour. This paper attempts to contribute to the endeavour of designing a universal model of vagueness in cooperative situations. Namely, some initial auxiliary steps towards the development of such a model are described. We use the concept of fuzzy coalitions suggested in [1], discuss the concepts of superadditivity and convexity, and introduce a concept of the coalitional structure of fuzzy coalitions.

The first version of this paper [10] was presented at the Czech-Japan Seminar in Valtice 2003. It was obvious that the roots of some open questions can be found in the concept of superadditivity (with consequences on some other related concepts), which deserve more attention. This version of the paper extends the previous one by discussion of alternative approaches to this topic.

## 1 Introduction

The classical mathematical model of cooperative behaviour, based on the concept of coalitional game, is deterministic. In this paper, we focus our attention on the transferable utility (TU) coalitional games (see, e. g., [9]). These games are characterized by a (non-empty and finite) set of players, which generates the class of admissible coalitions, and by total coalitional payoffs determining the common income of each coalition. Since knowledge of both these components (coalitions and payoffs) is only vague, a fuzzification of the original deterministic model appears desirable.

Two main approaches to the fuzzification of TU games can be found in the literature. Some authors deal with the fuzzification of coalitions. This approach is based on the idea that each player can participate in more than one coalition. The participation in particular coalitions can be of different degree, which influences also the distribution of coalitional payoff. This uncertainty is formally included into the model by fuzzification of coalitions – each coalition is a fuzzy subset of the set of all players, as shown, e. g., in [1, 2, 3] and also in [8]. The alternative approach to the fuzzification of TU games is concerned with the expected coalitional payoffs. This type of uncertainty can be represented by considering fuzzy payoffs (i.e., fuzzy numbers) instead of the traditional crisp characteristic function

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of the game. This model is described, e. g., in [4, 5, 7]. As the direct substitution of crisp payoffs by fuzzy values leads to some formal difficulties, as shown, e. g., in [4], it appears to be more adequate to transform this model into an alternative one where a fuzzy class of deterministic games is investigated instead of the fuzzy games (with fuzzy payoffs). In [6] and [7], we have shown that it is possible.

The presumed further research of fuzzy coalitional games could be oriented to the construction of a unitary model of uncertainty in cooperation including both of the sources of vagueness mentioned above. Recently, it has appeared useful to design theoretical tools for its development, among which the tools for handling coalitional structures, their fuzzy counterparts and their relevant properties play a significant role. In this contribution, we propose the concept of the fuzzy coalitional structure based on the model of fuzzy coalition from [1], and discuss the fuzzy analogies of superadditivity and convexity.

## 2 Crisp TU Coalitional Game

In the classical coalitional game theory (see e. g., [9] or introductory parts of [4]) the *game with transferable utility* (TU game) is defined as a pair  $(I, v)$  composed by the set of *players*  $I$  which is assumed to be non-empty and finite (without loss of generality we “name” the players by natural numbers and set  $I = \{1, 2, \dots, n\}$ ), and by the *characteristic function*  $v$ . If we call every set of players  $K \subset I$ , including  $K = I$  and the empty set  $K = \emptyset$ , a *coalition*, then the characteristic function  $v$  associates with every coalition  $K$  a real number  $v(K)$  which represents the common expected payoff of coalition  $K$ . We suppose that  $v(\emptyset) = 0$ . If, for each pair of disjoint coalitions  $K, K' \subset I$ ,

$$v(K \cup K') \geq v(K) + v(K') \quad (1)$$

then we say that the game is *superadditive*. If, for each  $K, K' \subset I$ ,

$$v(K \cup K') + v(K \cap K') \geq v(K) + v(K') \quad (2)$$

then the game is called *convex*. Every partition of  $I$  into disjoint coalitions, let us denote it  $\mathcal{K} = \{K_1, \dots, K_m\}, K_i \cap K_j = \emptyset$  for  $i \neq j, K_1 \cup \dots \cup K_m = I$ , is called a *coalitional structure*.

## 3 Fuzzy Coalitions

In the following sections, we denote for every set  $X$  by  $\mathcal{P}(X)$  the class of all subsets of  $X$ . We also accept the following notations – the symbol  $K$  (with eventual accents or indices) denotes the crisp coalitions from  $\mathcal{P}(I)$ .

Let us turn our attention to the fuzzification of coalitions. First we notice that each crisp coalition  $K$  can be identified with the vector

$$(\tau_K(1), \tau_K(2), \dots, \tau_K(n)), \quad (3)$$

where, for every player  $i \in I, \tau_K(i) = 1$  iff  $i \in K$  and  $\tau_K(i) = 0$  iff  $i \in I - K$ .

If we accept the assumption that the players participate in coalitions only with some part of their “power”, then we also accept that some players can participate in more than

one coalition. Such “partial” participation can be formalized by means of the fuzzy set theoretical tools (see [1, 2, 3]). Following Aubin [1], we define a *fuzzy coalition*  $L$  as a fuzzy subset of  $I$  with membership function  $\tau_L : I \rightarrow [0, 1]$ , where the value  $\tau_L(i) \in [0, 1]$  represents the degree in which player  $i \in I$  participates in fuzzy coalition  $L$ . The class of all fuzzy coalitions in the game  $(I, v)$  is denoted by  $\mathcal{F}(I)$  and we denote the fuzzy coalitions by (eventually accented or indexed) letters  $L$  or  $M$ . Evidently, every fuzzy coalition  $L$  can be identified with the  $n$ -dimensional vector

$$\tau_L = (\tau_L(1), \tau_L(2), \dots, \tau_L(n)), \quad \tau_L(i) \in [0, 1], \quad i = 1, 2, \dots, n. \quad (4)$$

It is natural to expect that fuzzy and crisp coalitions are somewhat mutually related. It turns out that the fuzzy coalitions may be considered as combinations of the cooperative endeavour of deterministic sets of players. Let us make this informal claim more precise.

If  $L_1, L_2, \dots, L_m$  are fuzzy coalitions represented by vectors

$$(\tau_{L_1}(1), \dots, \tau_{L_1}(n)), \quad (\tau_{L_2}(1), \dots, \tau_{L_2}(n)), \dots, (\tau_{L_m}(1), \dots, \tau_{L_m}(n))$$

and  $\lambda_1, \lambda_2, \dots, \lambda_m \in [0, 1]$  are real coefficients, then the real-valued vector

$$(\lambda_1 \tau_{L_1}(1) + \lambda_2 \tau_{L_2}(1) + \dots + \lambda_m \tau_{L_m}(1), \dots, \lambda_1 \tau_{L_1}(n) + \lambda_2 \tau_{L_2}(n) + \dots + \lambda_m \tau_{L_m}(n)) \quad (5)$$

is called a *combination* of coalitions  $L_1, L_2, \dots, L_m$ . In abbreviatory form, we write  $\lambda_1 L_1 + \lambda_2 L_2 + \dots + \lambda_m L_m$ . If, moreover,  $\lambda_1 + \lambda_2 + \dots + \lambda_m \leq 1$  then vector (5) is called a *subconvex combination* of  $L_1, L_2, \dots, L_m$ , and if the sum is equal to 1 then vector (5) is called a *convex combination* of  $L_1, L_2, \dots, L_m$ .

The *crisp coalitions* deserve a special attention. Obviously, they can be considered as special cases of fuzzy coalitions. Namely, they can be identified with those fuzzy coalitions whose membership functions may take only values 0 or 1. The  $n$  players of the game can form  $2^n$  crisp coalitions. To simplify the formulations of some statements, we set  $N = 2^n - 1$ .

Obviously each fuzzy coalition  $L$  characterized by vector  $\tau_L$  can be expressed as a combination of crisp coalitions. For example, setting  $\lambda_1 = \tau_L(1), \lambda_2 = \tau_L(2), \dots, \lambda_n = \tau_L(n)$ , we obtain coalition  $L$  as combination of crisp one-player coalitions  $\{1\}, \{2\}, \dots, \{n\}$ . Such a combination may be neither subconvex nor convex combination. In what follows, we are interested in the representation of fuzzy coalitions by means of subconvex combination or convex combinations of some crisp coalitions.

**Observation 1.** *Every fuzzy coalition can be represented by a convex combination of crisp coalitions.*

*Proof.* This is a direct consequence of the fact that every nonempty convex compact set in a finite dimensional space is a convex combination of its extreme points. Here the fuzzy coalitions are represented by points of the unit hypercube  $[0, 1]^n$ , and the crisp coalitions are represented by the vertices of that hypercube.  $\square$

The following example shows that a fuzzy coalition can be represented by more than one convex combinations of crisp coalitions.

**Example 1.** Let us consider a three-players set  $I = \{1, 2, 3\}$  and its fuzzy coalition  $L$  represented by the triple  $(\tau_L(1), \tau_L(2), \tau_L(3))$ , where  $\tau_L(1) = \frac{1}{3}$ ,  $\tau_L(2) = 1$ ,  $\tau_L(3) = \frac{1}{3}$ . Then it can be represented either by a pair of crisp coalitions  $(I, K)$ .  $\tau_I(1) = \tau_I(2) = \tau_I(3) = 1$  and  $\lambda_I = \frac{1}{3}$ ;  $K = \{2\}$ , i. e.,  $\tau_K(1) = \tau_K(3) = 0, \tau_K(2) = 1$  and  $\lambda_K = \frac{2}{3}$ . Or, it can be represented by a triple of crisp coalitions  $K_1, K_2, K_3 \in \mathcal{P}(I)$ , where  $K_1 = \{1, 2\}$ ,  $K_2 = \{2, 3\}$ ,  $K_3 = \{2\}$ , i. e.

$$\begin{aligned} \tau_{K_1}(1) = \tau_{K_1}(2) = 1, \quad \tau_{K_1}(3) = 0; \quad \tau_{K_2}(1) = 0, \quad \tau_{K_2}(2) = \tau_{K_2}(3) = 1; \\ \tau_{K_3}(1) = \tau_{K_3}(3) = 0, \quad \tau_{K_3}(2) = 1, \end{aligned}$$

and  $\lambda_{K_1} = \lambda_{K_2} = \lambda_{K_3} = \frac{1}{3}$ . □

Let us note that each fuzzy coalition  $L$  from  $\mathcal{P}(I)$ , can be expressed as a convex combination of all crisp coalitions  $K_0, K_1, \dots, K_N$ , i. e.,  $L = \lambda_0 K_0 + \lambda_1 K_1 + \dots + \lambda_N K_N$ , where we denote  $K_0 = \emptyset$  and where some of the coefficients  $\lambda_j$ ,  $j = 0, \dots, N$ , may vanish. It means that  $L$  is characterized by an  $(N + 1)$ -dimensional vector  $(\lambda_0, \lambda_1, \dots, \lambda_N)$ . Vice-versa, each convex combination of all crisp coalitions represents some fuzzy coalition. In other words, each non-negative vector  $(\lambda_0, \lambda_1, \dots, \lambda_N)$  whose components sum up to 1 represents certain fuzzy coalition. It follows from the previous example that the mapping between fuzzy coalitions and vectors  $(\lambda_j)_{j=0, \dots, N}$  is not one-to-one.

If, again following Aubin, we define a cooperative fuzzy game with transferable utility as a function  $w$  that assigns to every fuzzy coalition  $L$  a real number  $w(L)$  and satisfies the condition  $w(\emptyset) = 0$ , then it is natural to ask which of such fuzzy games can be considered as extensions of crisp games. It is rational to require that the payoffs to fuzzy coalitions of such extensions are related in some specific way to the payoffs of the corresponding crisp coalitions.

If a fuzzy coalition  $L$  is represented by a subconvex combination of crisp coalitions (5), then it is natural to consider the analogous subconvex combination of the payoffs

$$\lambda_1 v(K_1) + \lambda_2 v(K_2) + \dots + \lambda_m v(K_m). \quad (6)$$

As there may exist several representations of a fuzzy coalition, we define the value  $\bar{v}(L)$  for any fuzzy coalition  $L$  by

$$\begin{aligned} \bar{v}(L) = \sup (\lambda_1 v(K_1) + \lambda_2 v(K_2) + \dots + \lambda_m v(K_m) : (K_1, K_2, \dots, K_m), (\lambda_1 \lambda_2, \dots, \lambda_m) \\ \text{which are subconvex combinations representing } K). \end{aligned} \quad (7)$$

We already know that each fuzzy coalition  $L$  can be identified with a vector  $(\lambda_0, \lambda_1, \dots, \lambda_N) \in [0, 1]^N$  such that  $L = \lambda_0 K_0 + \dots + \lambda_N K_N$  is a convex combination of all crisp coalitions. The following Observation 2 shows that the class of all subconvex combination in (7) can be reduced to the class of convex combinations of all crisp coalitions.

**Observation 2.** *For every fuzzy coalition  $L$ , the value  $\bar{v}(L)$  is given by*

$$\begin{aligned} \bar{v}(L) = \sup (\lambda_1 v(K_1) + \lambda_2 v(K_2) + \dots + \lambda_N v(K_N) : (K_1, K_2, \dots, K_m), (\lambda_1 \lambda_2, \dots, \lambda_m) \\ \text{where } K = \lambda_0 K_0 + \dots + \lambda_N K_N \text{ is convex combination of all coalitions.} \end{aligned}$$

*Proof.* We are to show that any subconvex combination of an arbitrary set of crisp coalitions can be substituted by a convex combination of all crisp coalitions. This property

is obvious. If for some fuzzy coalition  $L$  and a set of crisp coalitions  $K_1, \dots, K_m$ , with coefficients  $\lambda_1, \dots, \lambda_m$ , the relation

$$K = \lambda_1 K_1 + \dots + \lambda_m K_m$$

is true, then we put  $\lambda_j = 0$  for  $j \notin \{1, \dots, m\}$ ,

$$L = \lambda_0 K_0 + \lambda_1 K_1 + \dots + \lambda_N K_N$$

and

$$\lambda_0 v(K_0) + \lambda_1 v(K_1) + \dots + \lambda_N v(K_N) = \lambda_1 v(K_1) + \dots + \lambda_m v(K_m),$$

where

$$\lambda_0 + \lambda_1 + \dots + \lambda_{N-1} = \lambda_1 + \dots + \lambda_m \leq 1.$$

If the combination in (7) is subconvex but not convex, then we may put  $\lambda_0 = 1 - (\lambda_1 + \dots + \lambda_m)$ , and the statement is proved, as  $v(K_0) = 0$ .  $\square$

In this way we have associated with every crisp game  $v$  exactly one fuzzy game  $w$ , namely  $w = \bar{v}$ .

**Observation 3.** *If  $v$  is the characteristic function of a deterministic TU game, then  $\bar{v}(K) = v(K)$  for every crisp coalition  $K$ .*

*Proof.* Let  $K$  be a crisp coalition. Since  $K$  can be represented by the convex combination with  $m = 1, \lambda_1 = 1, K_1 = K$ , we have  $\bar{v}(K) \geq v(K)$ . Now suppose that  $\bar{v}(K) > v(K)$ . Then there is a subconvex combination  $\lambda_1 K_1 + \dots + \lambda_m K_m$  of crisp coalitions such that

$$\lambda_1 v(K_1) + \dots + \lambda_m v(K_m) > v(K). \quad (8)$$

There is no loss of generality in assuming that all coefficient  $\lambda_j, j = 1, 2, \dots, m$ , are positive. It follows that  $K_j = K$  for each  $j = 1, 2, \dots, m$ . Indeed, if player  $i$  does not belong to coalition  $K$ , then

$$\lambda_1 \tau_{K_1}(i) + \dots + \lambda_m \tau_{K_m}(i) = 0, \quad (9)$$

which is possible only if  $\tau_{K_j}(i) = 0$  for each  $j = 1, 2, \dots, m$ . Now suppose that  $i \in K$  and  $i \notin K_j$  for some  $j$ . Let  $J$  be the set of all indices  $j$  for which  $i \notin K_j$ . Then we have

$$\sum_{j \in J} \lambda_j \tau_{K_j}(i) + \sum_{j \notin J} \lambda_j \tau_{K_j}(i) = 1,$$

which can be satisfied only if  $J$  is empty. To complete the proof, we note that, for  $K_1 = K_2 = \dots = K_m = K$ , we have

$$\lambda_1 v(K_1) + \dots + \lambda_m v(K_m) = (\lambda_1 + \dots + \lambda_m) v(K) \leq v(K),$$

which contradicts (8).

## 4 Convexity and Superadditivity

In this section we denote by  $M, L$  the fuzzy coalitions  $M = (\tau_M(1), \dots, \tau_M(n))$ ,  $L = (\tau_L(1), \dots, \tau_L(n))$  and, in accordance with the principles of the fuzzy set theory, we define their union and intersection by

$$\begin{aligned} M \cup L &= (\max(\tau_M(1), \tau_L(1)), \dots, \max(\tau_M(n), \tau_L(n))), \\ M \cap L &= (\min(\tau_M(1), \tau_L(1)), \dots, \min(\tau_M(n), \tau_L(n))). \end{aligned}$$

The fuzziness of coalitions leads to some difficulties if we are to consider their disjointness. As a consequence, there can be some methodological problems if we want to distinguish between superadditivity and convexity of games with fuzzy coalitions (see (1),(2)). There exist, evidently, several possible views on this topic. Here, we choose the following very simple one. In this section, we consider the fuzzy coalitional game  $(I, w)$  with fuzzy coalitions and with characteristic function  $w$  as introduced in Section 3 and constructed by means of (7).

We say that a game  $(I, w)$  with fuzzy coalitions is convex iff for any pair of fuzzy coalitions  $K, L$ , analogously to (2),

$$w(M \cup L) + w(M \cap L) \geq w(M) + w(L). \quad (10)$$

As the convexity does not require the disjointness of coalitions, there is no formal difficulty with its re-formulation in the environment of fuzzy coalitions.

**Observation 4.** *If  $(I, v)$  is a crisp convex TU game, then the game  $(I, \bar{v})$  with fuzzy coalitions is convex in the sense of (10), too.*

*Proof.* The statement follows from Remark 2, Observation 2 and from (7). If  $L$  is a convex combination of  $\{K_1^L, \dots, K_m^L\}$  and  $M$  is a convex combination of  $\{K_1^M, \dots, K_p^M\}$ , with some coefficients then  $L \cup M$  can be expressed as convex combination of  $\{K_1^L, \dots, K_m^L, K_1^M, \dots, K_p^M\}$  characterizes  $L \cup K$  with some coefficients. Due to the convexity of the original crisp game the desired inequality is true.  $\square$

To define the superadditivity of games with fuzzy coalitions, we first say that two fuzzy coalitions  $M, L$  are disjoint iff for all  $i \in I$ ,  $\min(\tau_M(i), \tau_L(i)) = 0$ . Then we say that a TU game  $(I, w)$  with fuzzy coalitions is superadditive iff for any pair of disjoint fuzzy coalitions  $K, L$

$$w(M \cup L) \geq w(M) + w(L). \quad (11)$$

The following statements are obvious.

**Remark 1.** If a TU game with fuzzy coalitions is convex then it is superadditive.

**Observation 5.** *If  $(I, v)$  is a superadditive crisp TU game, then  $(I, \bar{v})$  is superadditive in the sense of (11).*

*Proof.* The validity of this observation immediately follows from Remark 1, Observation 3 and from (10) and (11).  $\square$

The concept of superadditivity (and, in some sense, also convexity) presented in the previous section does not appear to be very satisfactory. The source of the doubts can be found in the condition of disjointness, formulated above.

The disjointness of fuzzy coalitions formulated by the condition

$$\min(\tau_M(i), \tau_L(i)) = 0 \quad \text{for all } i \in I$$

exactly copies the analogous concept in the deterministic game theory, and also its properties (see proof of Observation 4, Observation 5 and Remark 1) are sufficiently similar to the ones known in the deterministic theory. Nevertheless, its extreme form may be considered for too dogmatic in the case of the “weak” participation of players in fuzzy coalitions. Let us consider a modified concept of disjointness and check its basic properties. We say that fuzzy coalitions  $L, M$  are *weakly disjoint* if for each  $i \in I$

$$\tau_L(i) + \tau_M(i) \leq 1.$$

**Remark 2.** It is easy to see that any pair of disjoint fuzzy coalitions is also weakly disjoint, and that two crisp coalitions  $K, K'$  are weakly disjoint iff they are disjoint.

If we are to formulate the definition of superadditivity for weakly disjoint coalitions instead of the usual disjointness considered earlier in this section then we say that the fuzzy game  $(I, w)$  is *strongly superadditive* iff  $w(L \cup M) \geq w(L) + w(M)$  for each pair of weakly disjoint coalitions  $L$  and  $M$ .

**Remark 3.** Obviously, any strongly superadditive fuzzy game is superadditive in the above sense, and for disjoint crisp coalitions  $K, K'$ ,  $w(K \cup K') \geq w(K) + w(K')$  turns into (1) as follows from Observation 3.

Unfortunately, the implication between the convexity and superadditivity which we know from the deterministic case and also from Remark 1 is not generally guaranteed for the strong superadditivity, as shown in the next example.

**Example 2.** Let us consider 3-players coalitional game  $(I, v)$ ,  $I = \{1, 2, 3\}$ ,

$$\begin{aligned} v(\emptyset) &= 0, & v(\{1\}) &= v(\{2\}) = v(\{3\}) = 1, \\ v(\{1, 2\}) &= v(\{1, 3\}) = v(\{2, 3\}) = 2, & v(I) &= 3. \end{aligned}$$

This game is convex and superadditive in the deterministic sense, and, therefore, by Observation 4, the corresponding  $\bar{v}$  is also convex. Let us consider fuzzy coalitions in this game, and let us construct their characteristic function  $\bar{v}$  by means of (7). Let  $L, M$  be fuzzy coalitions, such that

$$\tau_M(1) = 1, \quad \tau_M(2) = \frac{1}{2}, \quad \tau_M(3) = 0, \quad \tau_L(1) = 0, \quad \tau_L(2) = \frac{1}{2}, \quad \tau_L(3) = 1.$$

We can see that coalitions  $M$  and  $L$  are weakly disjoint, and that

$$\tau_{L \cup M}(1) = \tau_{L \cup M}(3) = 1, \quad \tau_{L \cup M}(2) = \frac{1}{2}, \quad \tau_{L \cap M}(1) = \tau_{L \cap M}(3) = 0, \quad \tau_{L \cap M}(2) = \frac{1}{2}.$$

Furthermore, coalitions  $K, L, K \cup L, K \cap L$  can be expressed as convex combinations of crisp coalitions

$$\begin{aligned} M &= \frac{1}{2}\{1, 2\} + \frac{1}{2}\{1\}, & L &= \frac{1}{2}\{2, 3\} + \frac{1}{2}\{3\} \\ L \cup M &= \frac{1}{2}I + \frac{1}{2}\{1, 3\}, & L \cap M &= \frac{1}{2}\{2\} + \frac{1}{2}\emptyset. \end{aligned}$$

Then it is possible to verify that, according to (7),

$$\bar{v}(M) = \frac{3}{2}, \quad \bar{v}(L) = \frac{3}{2}, \quad \bar{v}(L \cup M) = \frac{5}{2}, \quad \bar{v}(L \cap M) = \frac{1}{2},$$

which means that

$$\bar{v}(L \cup M) < \bar{v}(M) + \bar{v}(L)$$

where  $M$  and  $L$  are weakly disjoint. Thus the convex game  $\bar{v}$  is not strongly superadditive.

## 5 Coalitional Structures

The concept which can be especially significant for the further processing of TU games with fuzzy coalitions, namely for the definition of their core, is the concept of coalitional structure. In our model, the coalitional structure is defined as any class of fuzzy coalitions  $\mathcal{L} = \{L_1, L_2, \dots, L_m\}$  such that, for all players  $i \in I$ ,

$$\tau_{L_1}(i) + \tau_{L_2}(i) + \dots + \tau_{L_m}(i) = 1. \quad (12)$$

It is easy to verify the validity of the following statements.

**Remark 4.** If the coalitions  $L_1, L_2, \dots, L_m$  in the above definition are crisp then they form a coalitional structure in the deterministic sense of Section 2.

**Observation 6.** If a TU crisp game  $(I, v)$  is such that  $(I, \bar{v})$  is superadditive in the sense of (11) and  $\mathcal{L} = \{L_1, L_2, \dots, L_m\}$  is a coalitional structure, then

$$\bar{v}(I) \geq \bar{v}(L_1) + \bar{v}(L_2) + \dots + \bar{v}(L_m).$$

*Proof.* Each fuzzy coalition  $L_j$  from the structure  $\mathcal{L}$  can be represented by some class of crisp coalitions  $K_1^{(j)}, \dots, K_{p_j}^{(j)}$  with coefficients  $\lambda_1^{(j)}, \dots, \lambda_{p_j}^{(j)}$ . From Observation 2, we obtain  $\bar{v}(K_k^{(j)}) = v(K_k^{(j)})$ . Due to the finiteness of  $I$ , we may substitute the supremum in (7) by maximum, and we may assume, without loss of generality, that  $K_1^{(j)}, \dots, K_{p_j}^{(j)}$  is the very crisp representation of  $L_j$  for which also  $\bar{v}(L_j) = \lambda_1^{(j)}v(K_1^{(j)}) + \dots + \lambda_{p_j}^{(j)}v(K_{p_j}^{(j)})$ . It means that for every player  $i$

$$\tau_j(i) = \lambda_1^{(j)}\tau_{K_1^{(j)}}(i) + \dots + \lambda_{p_j}^{(j)}\tau_{K_{p_j}^{(j)}}(i)$$

and the definitoric property (12) implies

$$\sum_{j=1}^m \left( \lambda_1^{(j)}\tau_{K_1^{(j)}}(i) + \dots + \lambda_{p_j}^{(j)}\tau_{K_{p_j}^{(j)}}(i) \right) = 1.$$

Then the superadditivity of  $(I, \bar{v})$  implies.

$$\bar{v}(I) = v(I) \geq \sum_{j=1}^m \sum_{k=1}^{p_j} \lambda_k^{(j)} \left( v(K_k^{(j)}) \right) = \sum_{j=1}^m \bar{v}(L_j). \quad (13)$$

□

**Remark 5.** Observations 4 and 5 imply the validity of (13) even if  $(I, v)$  is superadditive.

## 6 Monotonicity of Payoffs

Dealing with the topic of the fuzzy coalitions and their forming, it can be useful to mention, at least briefly, the fundamental approach to the fuzzy coalitional pay-offs represented by the characteristic function. In this section, we consider a coalitional game  $(I, v)$  with fuzzy coalitions, and with characteristic function  $w$  assigning to each fuzzy coalition  $K$  its (crisp) pay-off  $w(K)$ . Some of those fuzzy coalitions are crisp, as mentioned in Section 3, and then there may (but need not) exist some relation between crisp and fuzzy coalitions, like (7).

In the previous sections, processing the concepts of superadditivity and convexity, we have respected the classical paradigm of additivity of pay-offs. In the case of deterministic coalitions, it is very natural, and also for the fuzzy coalitions it does not cause immediate problems, as we could see in Section 4. However, certain irregularities in the relation between convexity and superadditivity (cf. Example 2) evoke the question, if the additivity paradigm is adequate to the vague character of cooperation in fuzzy coalitions. Some applications of fuzzy set theory were based on an alternative paradigm, namely on the monotonicity principle. Let us test, at least very briefly, the behaviour of superadditivity and convexity based on the monotonous characteristic function.

We suppose that our game  $(I, w)$  with fuzzy coalitions fulfils the following two properties for any fuzzy coalitions  $L, M$ :

$$w(\emptyset) = 0, \quad (14)$$

where  $\emptyset$  is the empty (crisp) coalition ( $\tau_\emptyset(i) = 0$  for all  $i \in I$ ),

$$\begin{aligned} & \text{if } M \supset L \text{ then } w(M) \geq w(L) \\ & \text{where } M \supset L \text{ means that } \tau_M(i) \geq \tau_L(i) \text{ for all } i \in I. \end{aligned} \quad (15)$$

In such game, we may define the modified concepts of superadditivity and convexity as follows.

We say that  $(I, w)$  is *m-convex* iff for each pair of fuzzy coalitions  $L, M$ ,

$$\max [w(L \cup M), w(L \cap M)] \geq \max [w(M), w(L)], \quad (16)$$

where  $L \cup M$  and  $L \cap M$  are defined in Section 4.

We say that  $(I, w)$  is *m-superadditive* iff for any pair of weakly disjoint coalitions  $L, M$  (see Section 4),

$$w(L \cup M) \geq \max[w(M), w(L)]. \quad (17)$$

The monotonicity condition (15), however natural it is (each fuzzy coalition can earn at least as much as any smaller group of its members), appears to be strong regarding convexity and superadditivity.

**Observation 7.** Each game with fuzzy coalitions fulfilling (14) and (15) is m-superadditive and m-convex.

*Proof.* Due to (15),  $w(L \cup M) \geq w(L \cap M)$ ,  $w(L \cup M) \geq w(M)$  and  $w(L \cup M) \geq w(L)$ . These inequalities immediately prove the statement.  $\square$

**Remark 6.** Evidently, if a game  $(I, w)$  with fuzzy coalitions fulfils (15) then  $w(I) \geq w(L) \geq 0$  for crisp  $I$  and any fuzzy coalition  $L$ .

The above remark opens interesting possibilities for the eventual development of the core-like solution concept in games modeled on the monotonicity principle.

## 7 Concluding Remarks

The limited extend of this contribution admits to mention only the main ideas regarding the fuzziness of coalitional cooperation. Even this brief presentation of the topic shows that many other problems related to this concept become urgent. For example:

- The algorithm for computation of coefficients mentioned in the proof of Observation 3 and implicitly assumed in Remark 4 or in the proof of Observation 5.
- The method of quantitative measurement of “distance” between fuzzy coalitions.
- More advanced analysis of the superadditivity and convexity.

The solution of these problems may open the way to the natural transformation of the presented model of fuzzy coalitions to the fuzzy classes of crisp TU games, analogously to the fuzzification of payoffs suggested in [7].

It seems that the model of cooperation in fuzzy coalitions based on the paradigm of monotonicity deserves more attention for its respect to the specific features of vague structures of coalitions.

## References

- [1] J. P. Aubin: Cooperative fuzzy games. *Math. Oper. Res.* 6 (1981), 1–13.
- [2] B. Butnariu: Fuzzy games: a description of the concept. *Fuzzy Sets and Systems* 1 (1978), 181–192.
- [3] D. Butnariu, E.P. Klement: Triangular Norm-Based Measures and Games With Fuzzy Coalitions. Kluwer, Dordrecht 1993.
- [4] M. Mareš: Fuzzy Cooperative Games. Physica–Verlag, Heidelberg 2001.
- [5] M. Mareš, M. Vlach: Linear coalitional games and their fuzzy extensions. *Int. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*. Vol. 9 (2001), 3, 341–354.
- [6] M. Mareš: Fuzzy TU games and their classes. *Bulletin of the Czech Econometric Society*. Vol.7, Issue 13(2001), 83–88.

- [7] M. Mareš, M. Vlach: Fuzzy classes of cooperative games with transferable utility. *Scientiae Mathematicae Japonicae* 2 (2004), Vol. 60, 269–278.
- [8] I. Nishizaki, M. Sakawa: Fuzzy and Multiobjective Games for Conflict Resolution. Physica-Verlag, Heidelberg 2001.
- [9] J. Rosenmüller: The Theory of Games and Markets. North Holland, Amsterdam 1982.
- [10] M. Mareš, M. Vlach: Fuzzy coalitional structures. In: V. Novák, J. Ramík (eds.): Transactions of the 7th Czech–Japan Seminar on Data Processing and Decision-Making Under Uncertainty, Valtice 2003. Technical University, Ostrava 2003, 51–57.