Ultrasound Transmission Tomography Using Algebraic Reconstruction Techniques

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DAR 2006

Motivation

The problem

- having large overdetermined linear system $Ax = b$, find a solution minimising $||Ax - b||$
- the matrix is sparse, however the number of equation can be large (hundreds of thousand)

The Methods

- classical approach:
  - Least Mean Squares
  - Minimisation Techniques
- alternative: *Kaczmarz Method*

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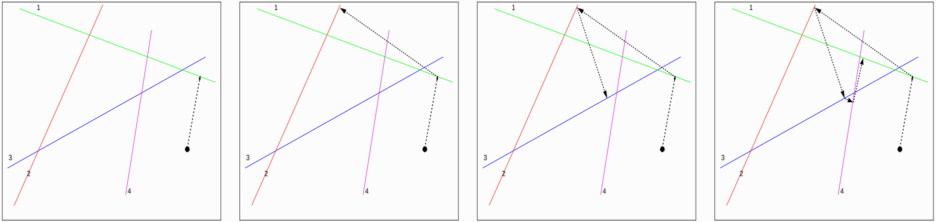
Kaczmarz Method

Principles

- geometrically, the system with *n* equation and *m* variables represents *n* hyperplanes in *m*-dimensional space
- the solution is a point in the space
- having an initial estimation, orthogonal projections onto the hyperplanes are performed (inner iteration)
- the process is iterated (outer iteration)



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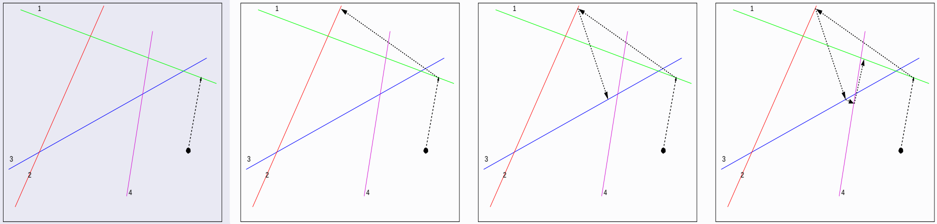

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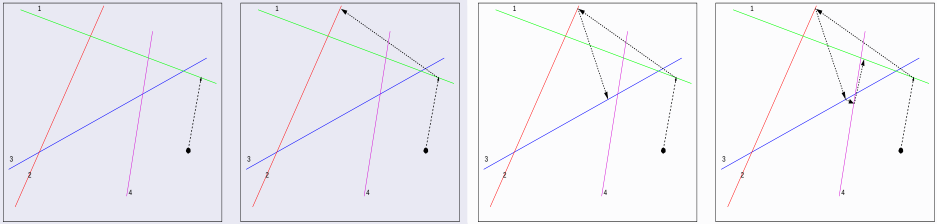

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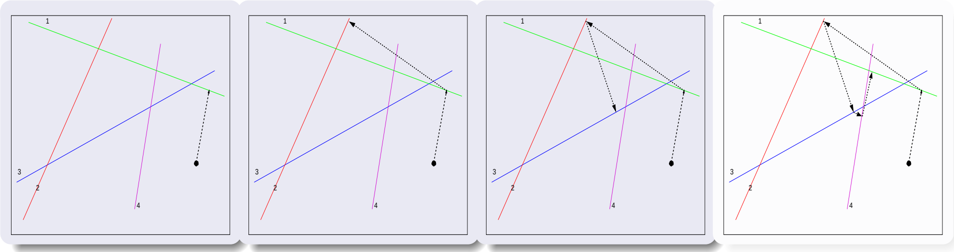

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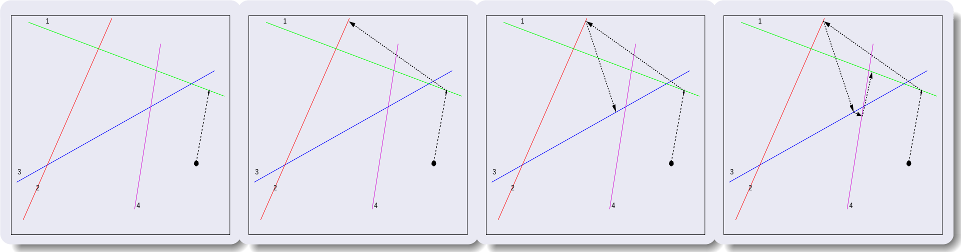
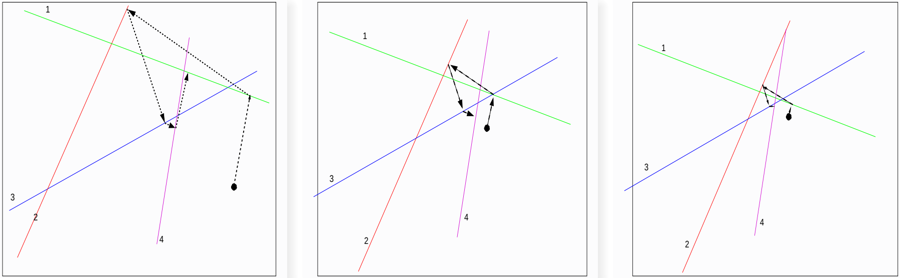


Extended Kaczmarz Method

Need for Improvement

- the classical Kaczmarz method does not converge every time to the least square solution
- extension: in each outer iteration, first the right-hand side vector is corrected
- the correction is performed as orthogonal projection of the initial RHS onto the columns



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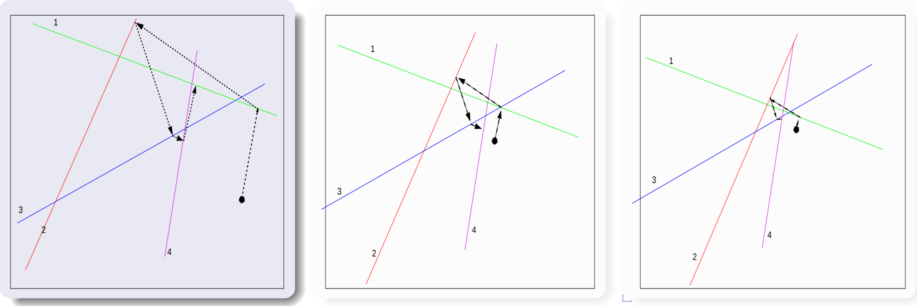

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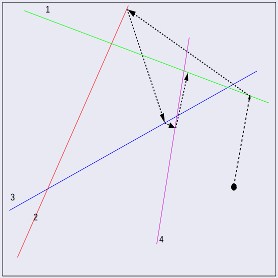

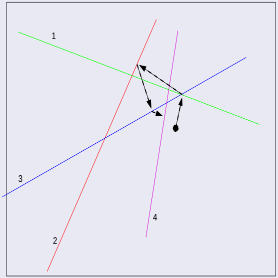

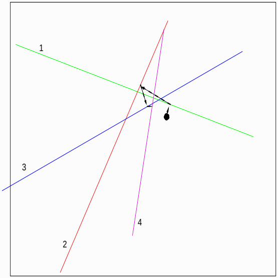

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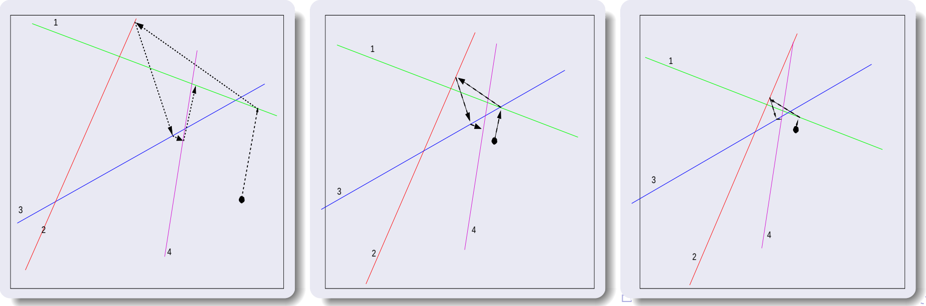

Original vs. Extended

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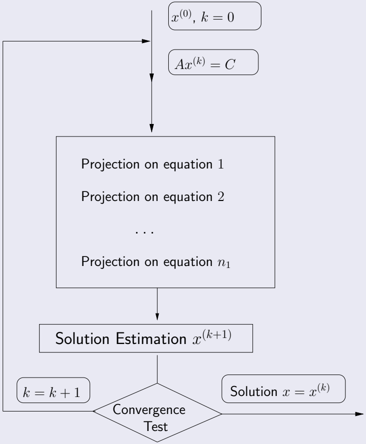

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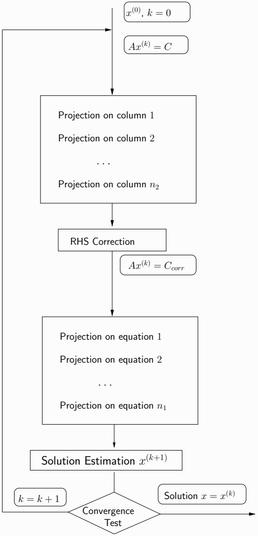

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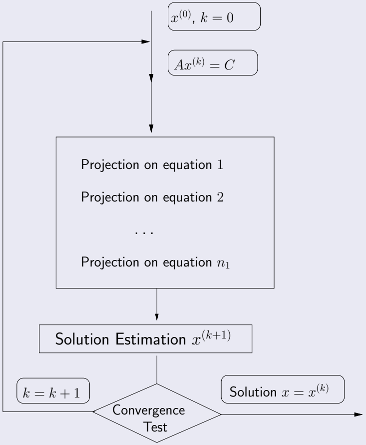

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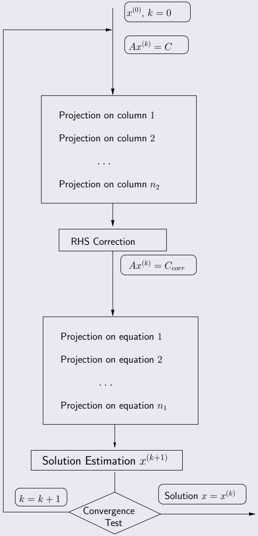
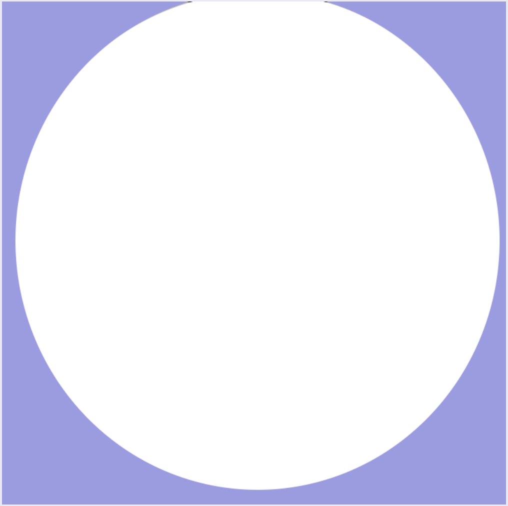


Reconstructed Images

Synthetic Data



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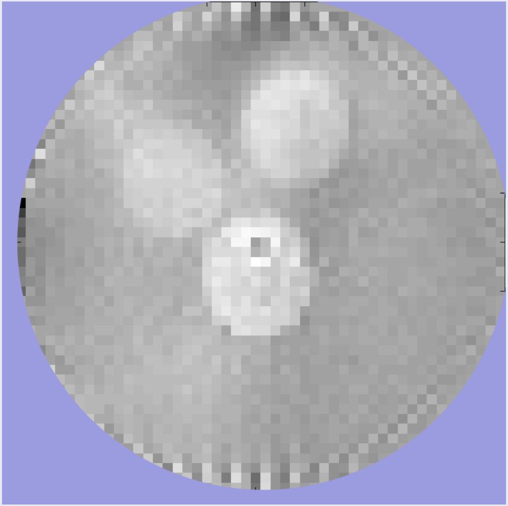

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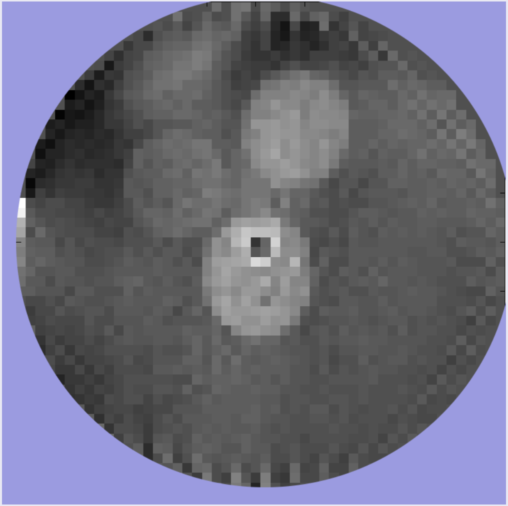

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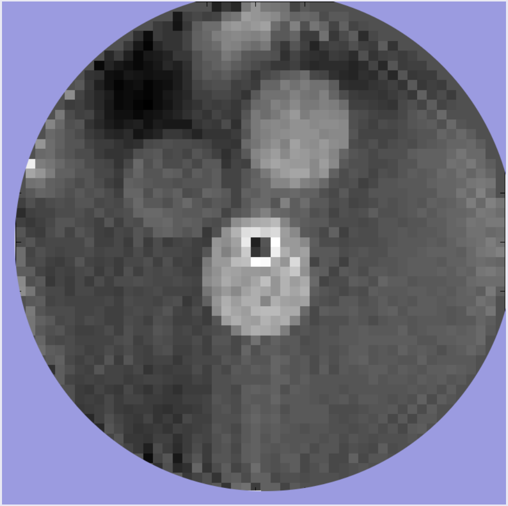

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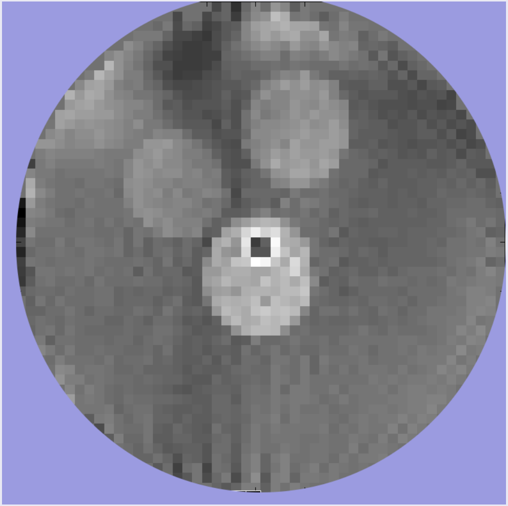

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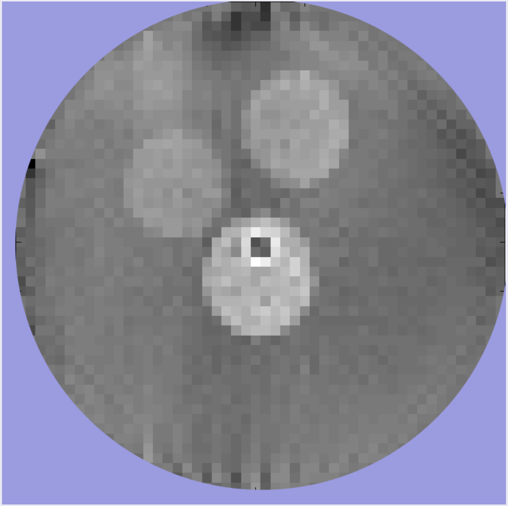

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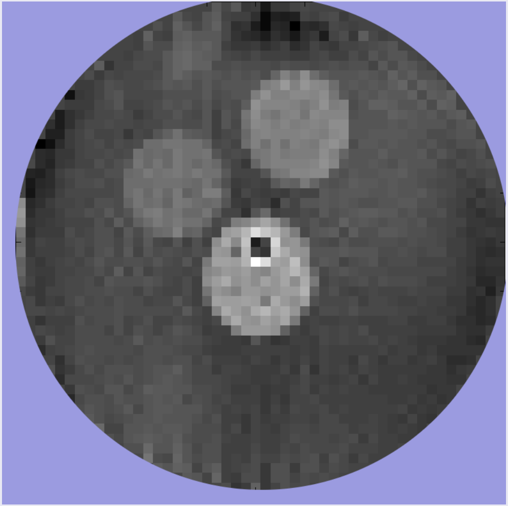

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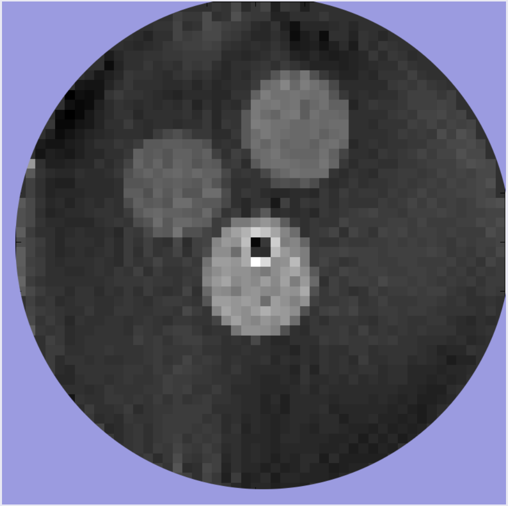

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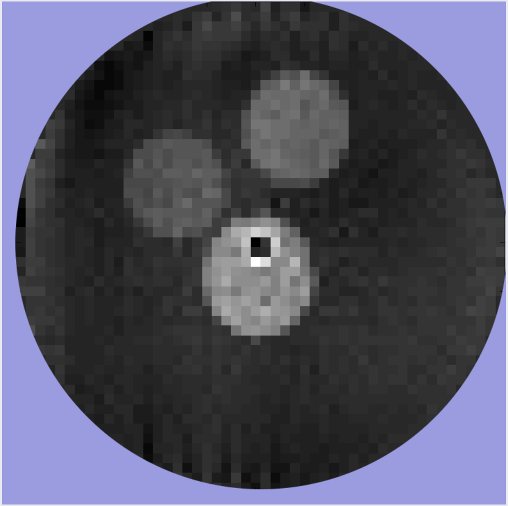

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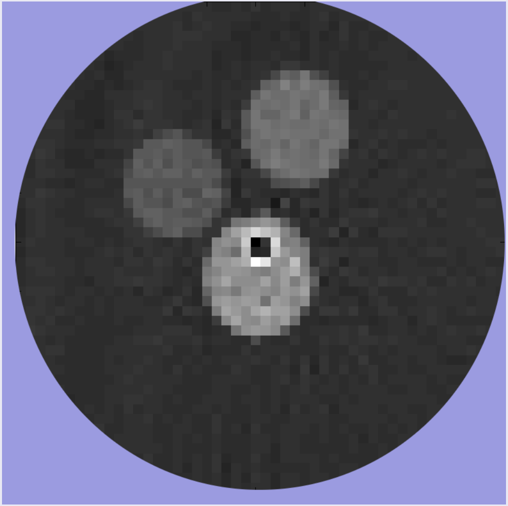

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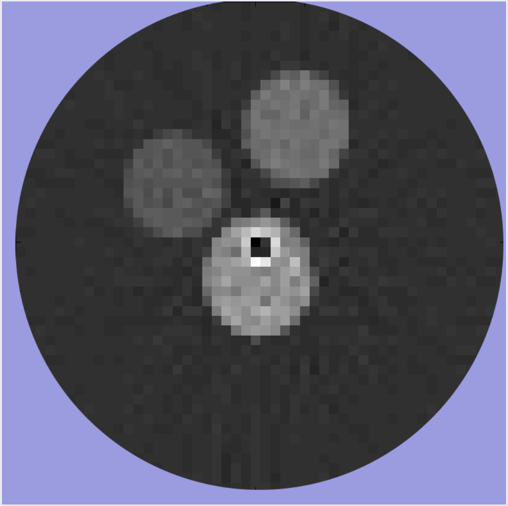

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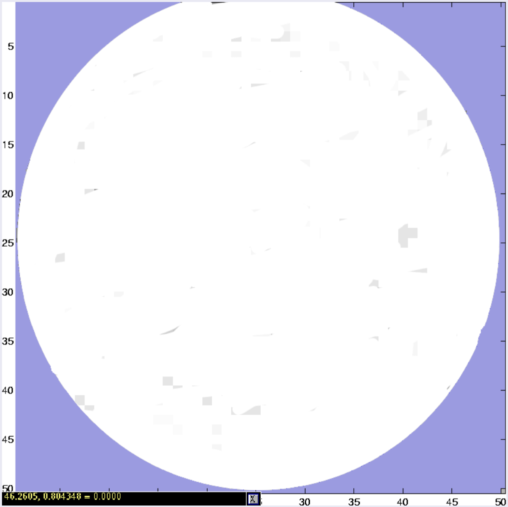

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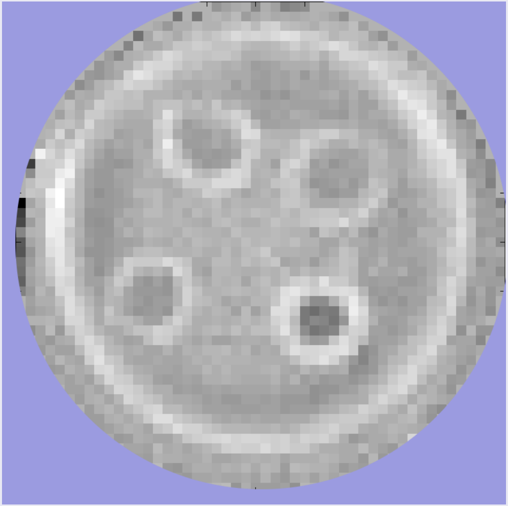

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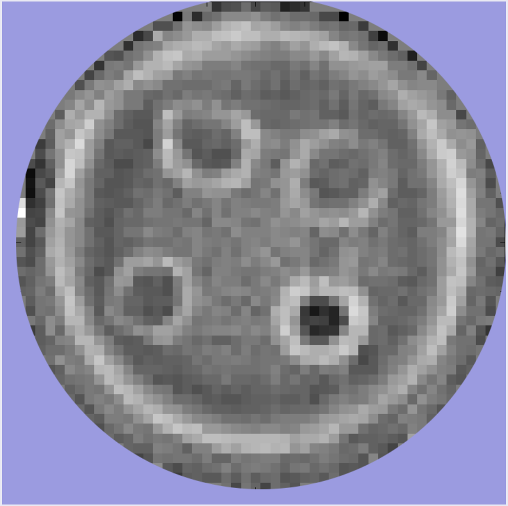

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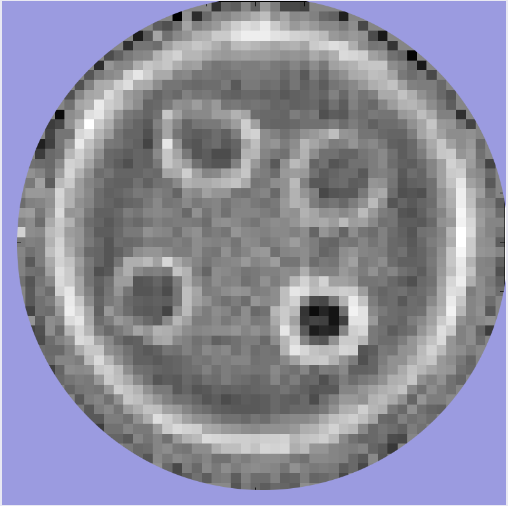

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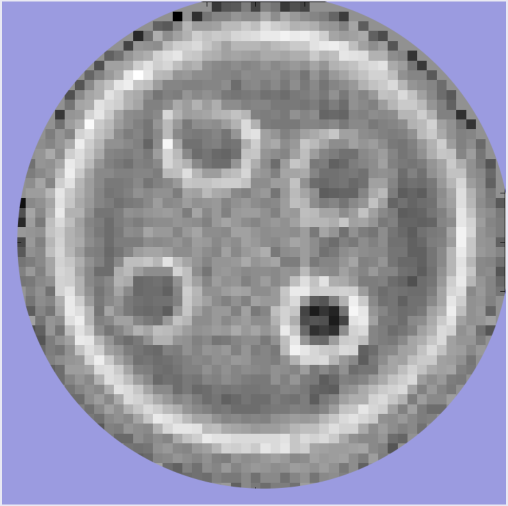

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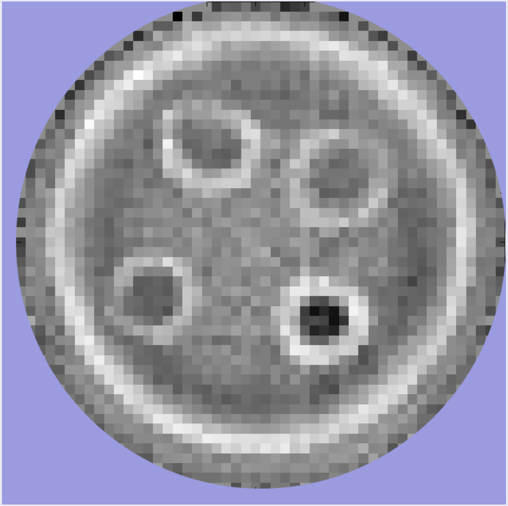

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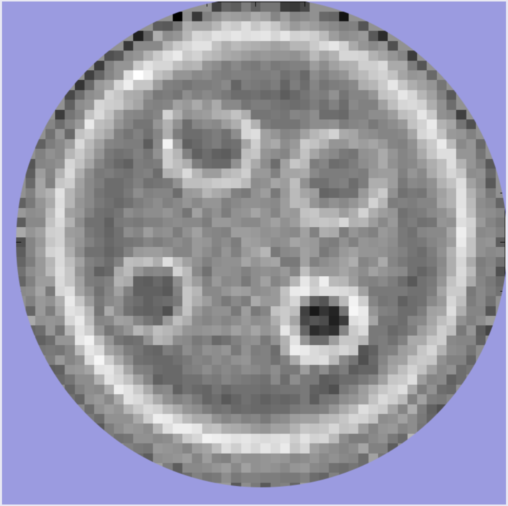

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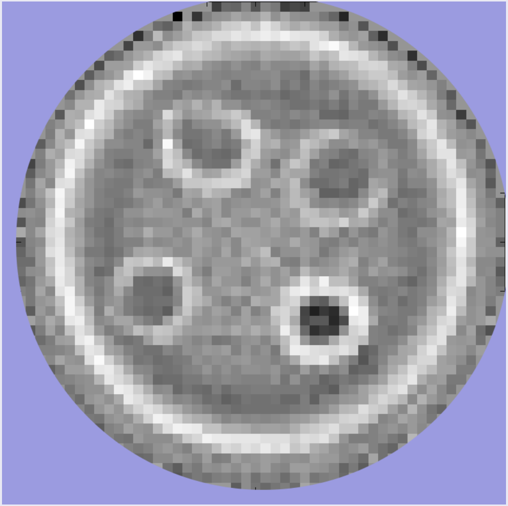

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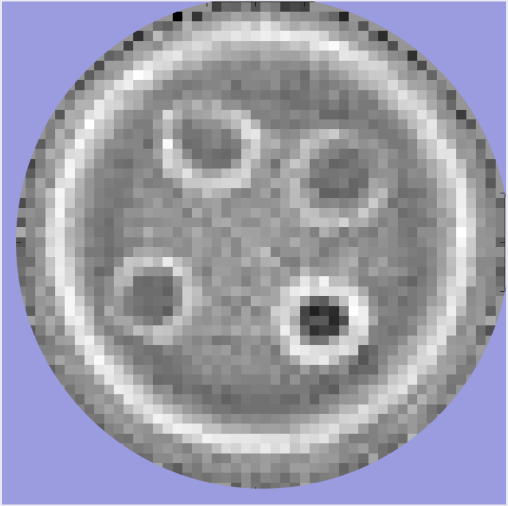

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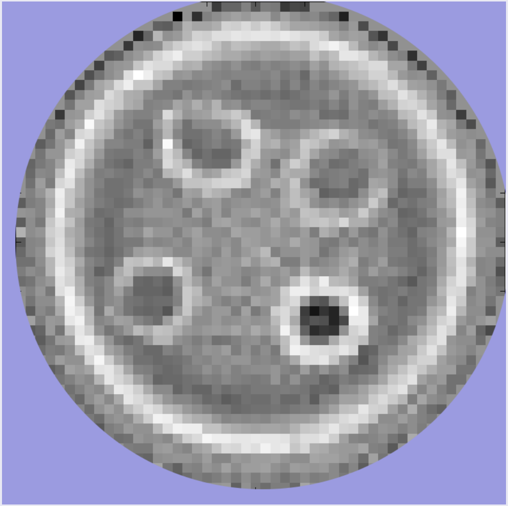

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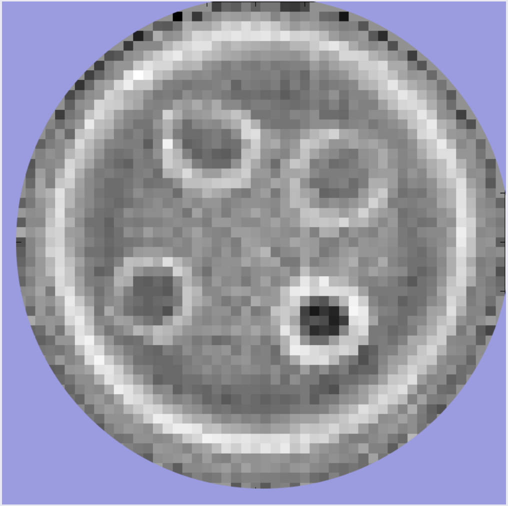
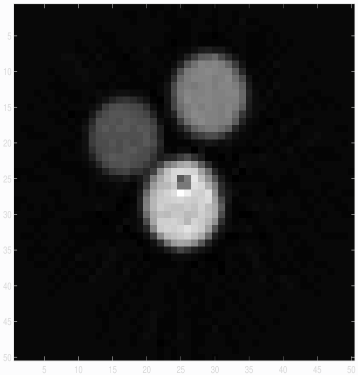
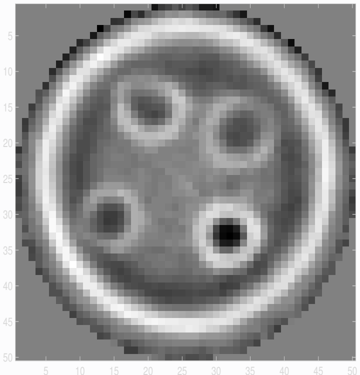


Further Improvement

Regularisation Technique

- usual technique for “smoothing” the solution
- special square matrix added to the system
- directly incorporated into the Kaczmarz method (adding *m* equations)





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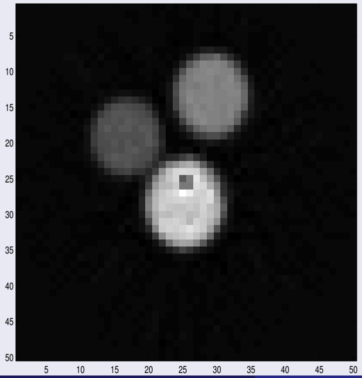

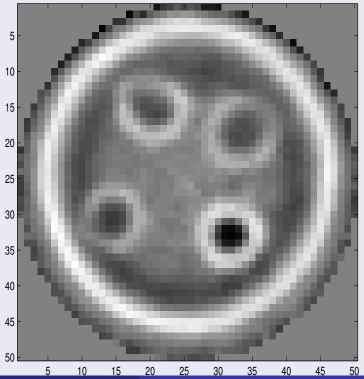

Partitioning Scheme

The Idea

- both the original and extended method are strictly sequential, since the computation in inner k-th iteration depends on the $(k-1)$-th inner iteration
- the straight forward possibility is to partition the matrix into blocks, which are then processed separately (in parallel)
- partitioning is static and regular

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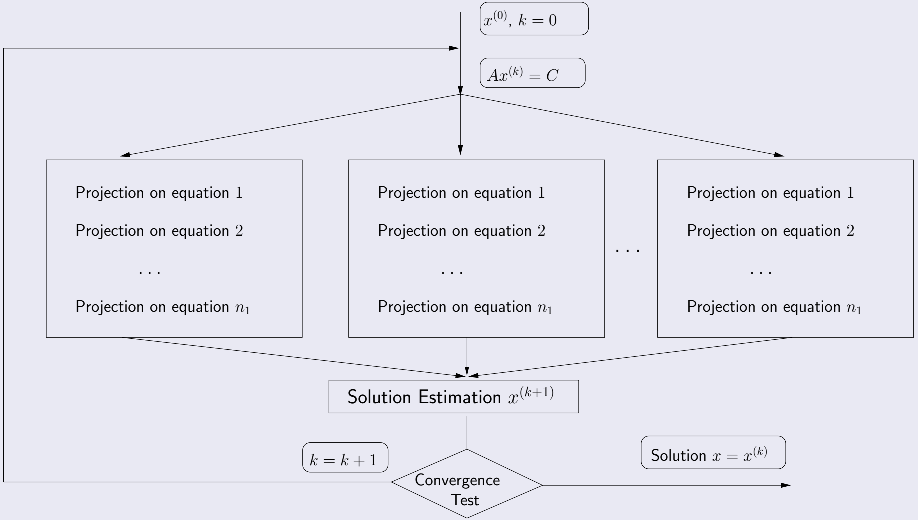
Partitioned Kaczmarz



Partitioned Extended KM

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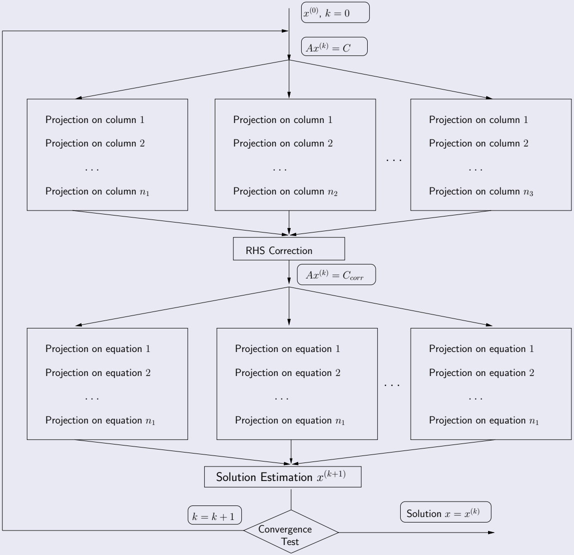

Measurements

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- the experiments were performed on both synthetic (127400 eq.) and phantom data (81000 eq)
- since the partitioning modifies the inner data-dependency inside the method, accuracy and convergence were analysed as well

Accuracy

- the accuracy was measured by the residual $r = ||Ax - b||$ which proved to be experimentally equivalent to the image difference $\Delta(a, b) = \sqrt{\sum_m \sum_n [a_{ref}(m, n) - a(m, n)]^2}$.
- the stopping criterium is given by relative residual $r^{(k)}/r^{(k-1)}$

Convergence

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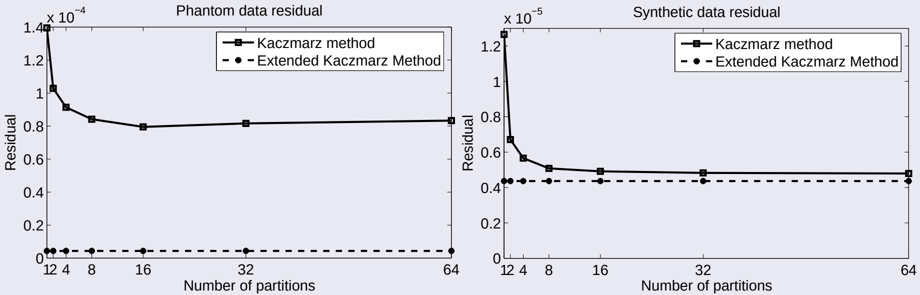
Number of partitions versus the residual



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- the accuracy of the extended method remains invariant

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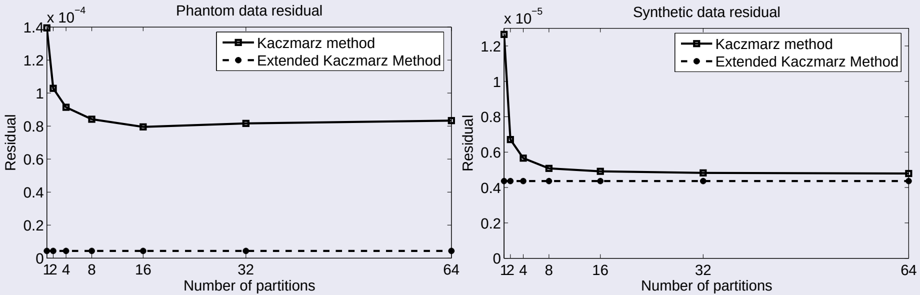

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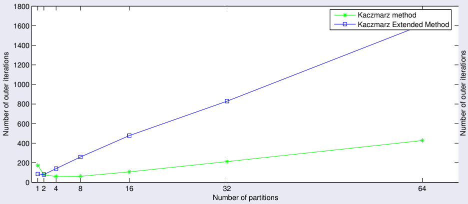
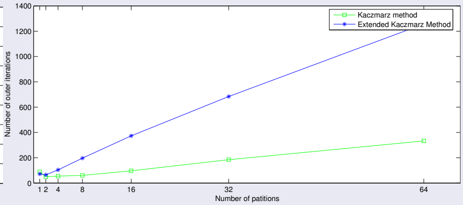
Number of Iterations vs. Partitioning





Analysis

- the convergence depends on the number of the partitions
- original method: for 2,4,8,16 —
  the convergence get better, for larger number of the partitions, increasing number of the outer iteration
- extended method: only for 2 is better, for larger number of the partitions get still worse

Convergence

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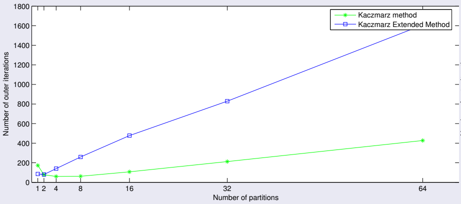

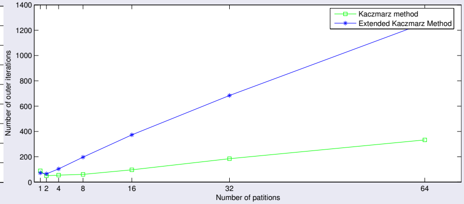

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Parallel Efficiency of Original Method

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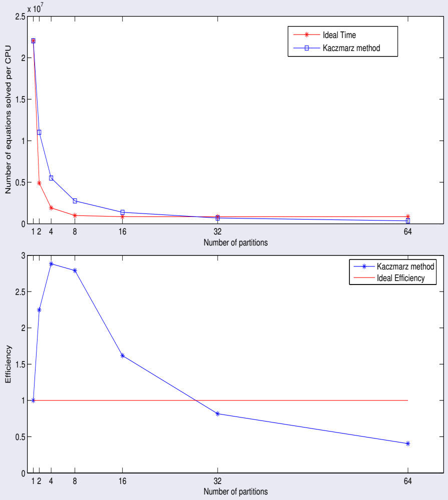

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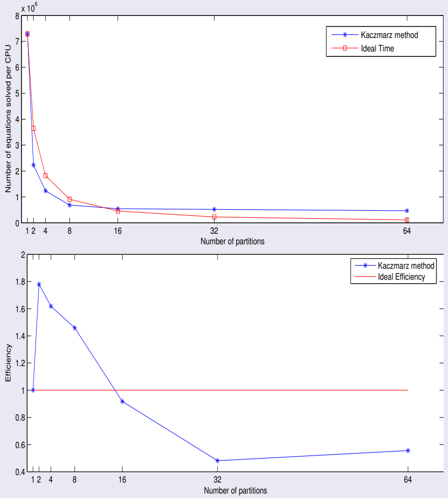

Parallel Efficiency of Extended Method

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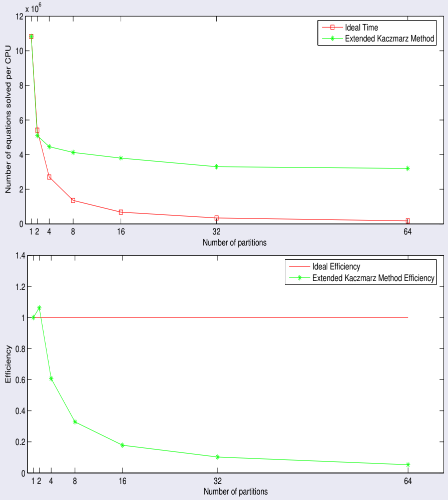

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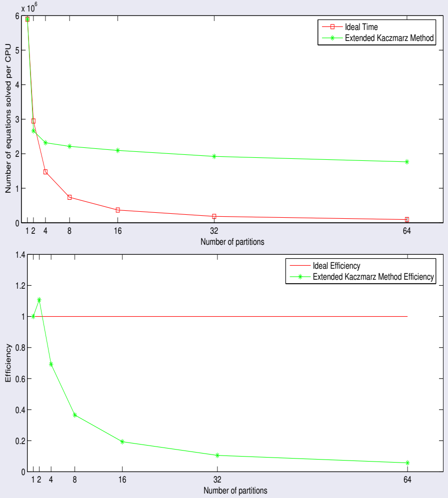

Summary

Conclusion

- comparison of the original and the extended version: the extended do better job
- partitioning scheme —
  makes sense for the original method:
  - phantom data: max 15× speed-up with 16 blocks (on 16 CPUs)
  - synthetic data: max 28× speed-up with 16 blocks (on 16 CPUs)
- not too much for the extended:
  - phantom data: max 2.3× speed-up with 4 blocks (on 4 CPUs)
  - synthetic data: max 2.6× speed-up with 4 blocks (on 4 CPUs)
- when the accuracy of the original method is sufficient, then the parallelisation can be applied with this method

Future Work

- larger set of equations (including the reflected signals)
- relaxation and regularisation extensions

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- partitioning scheme —
  makes sense for the original method:
  - phantom data: max 15× speed-up with 16 blocks (on 16 CPUs)
  - synthetic data: max 28× speed-up with 16 blocks (on 16 CPUs)
- not too much for the extended:
  - phantom data: max 2.3× speed-up with 4 blocks (on 4 CPUs)
  - synthetic data: max 2.6× speed-up with 4 blocks (on 4 CPUs)
- when the accuracy of the original method is sufficient, then the parallelisation can be applied with this method

Future Work

- larger set of equations (including the reflected signals)
- relaxation and regularisation extensions