ALTERNATIVE REPRESENTATION OF FUZZY COALITIONS*

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Abstract. The theory of cooperative games with vague cooperation is based on modelling fuzzy coalitions as fuzzy subsets of the set of all players who participate in the coalitions with some part of their "power". Here, we suggest an alternative approach assuming that coalitions are formed by relatively compact groups of individual players each of which represents a specific common interest. Each individual player may participate in several such groups and, as their member, in several coalitions. Our aim is to show that such an alternative model of fuzzy coalitions, in spite of its seemingly higher complexity, offers an interesting more sophisticated reflection of the structure of vague cooperation and of relations being in its background.

1 Introduction

The presented paper continues and develops the investigation partly described in [12] and motivated by some previous attempts to analyze fuzzy coalitions (and their pay-offs) as extensions of the crisp ones (cf. [11, 12] and also [14], e. g.).

The theory of fuzzy cooperative games has been developed since the seventieths of last century (see, e. g. [1, 2]). For recent results see, for example, [3, 4]. It is possible to say that some preliminary results, even if not formulated in terms of fuzzy sets but dealing with the participation of a player in more coalitions, were presented relatively early, see [6].

The traditional approach to modelling vague coalitions in cooperative games with transferable utility (TU-games) is to model fuzzy coalitions as fuzzy subsets of the set of all player (see [1, 2]). There is a weak, if any, connection between the game with such

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fuzzy coalitions and some deterministic TU-game whose crisp coalitions can be considered for some sort of patterns simply extended to fuzzy coalitions.

Trying to establish relations between the pay-offs in crisp coalitions and their fuzzy extensions, we have previously shown (see [10, 11]) that fuzzy coalitions can be well characterized by convex combinations of crisp coalitions. It is not difficult to think about the coefficients of these convex combinations as values of some membership function and, consequently, to represent a fuzzy coalition by a fuzzy subset of the set of all crisp coalitions. This procedure may appear to be rather complicated; instead of two levels of "interventions" into the game – the individual and the (fuzzy) coalitional – we consider three such levels – the individuals, the activity of (may be small) crisp coalitions, and the fuzzy coalitions resulting from the vague cooperation of those crisp groups. This extension of the model is not an end in itself. In the following sections, we are going to show that it leads to a richer and better structured model of the pattern of cooperative relations in the modelled environment. Moreover, it avoids some formal complications connected with fuzzification (and its interpretation) of some game theoretic concepts.

We focus our attention on a fully consistent model in which the fuzziness of some coalitions is represented by fuzzy sets of some groups of players which themselves are formed by members of identical interests. Each player can participate in several such groups and these groups can act in several coalitions (with different intensity). We call such groups "blocks" and their interpretation is quite natural: The participants of many social or economic processes act, quite frequently, as members of blocks whose preferences are homogeneous with regard to the global environment of the model (such blocks are, e. g., families, small firms, sport clubs, citizen activities, etc.). These blocks, "pay" some endeavour and share eventual profit or loss. The behavior of blocks as well as their approach to the total utility following from the cooperation, in displays in many practical situations some features of fuzzy systems rather than of deterministic or probabilistic ones. As the blocks represent rather close points of view (political, cultural in very wide sense, macroeconomic, etc.) than immediate profit, they can be considered for strictly defined, in our terminology crisp, coalitions. The vagueness of the level of cooperation, represented by the fuzziness of coalitions, is reflected rather by the degree of the participation of blocks in (more or less vaguely) cooperating units – called fuzzy coalitions.

From the methodological point of view, the model proposed below should respect several basic paradigms of the cooperative games theory and find a rational balance between their demands. Heuristically formulated, they may be summarized as follows:

— Every player aims to maximize his individual income; even his endeavour to maximize the total profit of some group is motivated by the goal to achieve maximal individual share.

— The players are motivated to cooperate if the cooperation does not decrease their individual incomes.

— The same is true for coalitions which are motivated to unite if it does not decrease their total income (i. e., also the individual incomes of their members.)

— On the other hand, the blocks are in much higher degree compact and more sharply determined. They form (fuzzy) coalitions with other blocks.

— Even if a player may be (and usually is) member of several blocks, he (even if he consequently follows his individual preferences) in the frame of each block acts in accordance with the behaviour of the block and respects the common degree of participation of each block in coalitions.

— Each block may participate in several coalitions.

Our aim is to establish and briefly discuss basic properties of the proposed model, and to show how the model is related to some already known fuzzy cooperative games. It turns out that despite the model takes its origin in the fuzzy coalitional relations, it leads to concepts which have already been investigated in the theory of TU-games with crisp coalitions and fuzzy pay-offs.

2 Fuzzy Coalitions as Fuzzy Sets of Blocks

The presented model stems from the traditional pattern of deterministic (or, crisp) coalitional *game with transferable utility* (briefly TU-game) defined as a pair (I, v) where I is the set of *players* and v is the *characteristic function* that assigns to each subset K of I a real number $v(K)$ and where $v(\emptyset) = 0$ for the empty subset of I (cf. [5, 13]). Without significant loss of generality, we suppose that $v(K) \geq 0$ for all $K \subset I$. To simplify notations, we "name" the players by the first positive integers, that is, if there are n players then $I = \{1, 2, \ldots, n\}$.

The crisp subsets of I can be intepreted as in some sense pure coalitions, not "softened" by uncertainty. In the following paragraphs, they are called *blocks*.

Throughout the paper, if M is a set then we denote by $\mathcal{F}(M)$ the set of all fuzzy subsets of M, and by $\mathcal{P}(M)$ the set of all crisp subsets of M.

The set of all blocks will be denoted by $\mathcal{P}(I)$. An alternative notation for the set of all blocks will be $\{K_0, K_1, \ldots, K_N\} = \mathcal{P}(I)$ where $K_0 = \emptyset$ and $N = 2^n - 1$. Generally, we use (maybe somehow indexed or accented) letter K (or K_j, $K', K^*, \ldots$) to denote particular blocks from $\mathcal{P}(I)$.

Every fuzzy subset $\mathcal{L}$ of $\mathcal{P}(I)$, $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$, with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0, 1]$ is called a *fuzzy coalition*. In accordance with the usual fuzzy set theoretical notation, we define the union and intersection of fuzzy coalitions $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ as fuzzy coalitions with

$$(1) \qquad \beta_{\mathcal{L}\cup\mathcal{M}}(K) = \max\left(\beta_{\mathcal{L}}(K),\, \beta_{\mathcal{M}}(K)\right), \quad \beta_{\mathcal{L}\cap\mathcal{M}}(K) = \min\left(\beta_{\mathcal{L}}(K),\, \beta_{\mathcal{M}}(K)\right)$$

for $K \in \mathcal{P}(I)$.

The concept "fuzzy coalition" as used in most of this paper, is defined as an element of $\mathcal{F}(\mathcal{P}(I))$, whereas in the referred literature (including the following Intermezzo 1) this term is used for elements of $\mathcal{F}(I)$. To distinguish between these different approaches also typographically, we denote the fuzzy classes of blocks from $\mathcal{F}(\mathcal{P}(I))$ by letters $\mathcal{L}$, $\mathcal{M}, \ldots$, and the fuzzy sets of players from $\mathcal{F}(I)$ by italic letters L, $M, \ldots$.

Let X be an arbitrary set. If A is a subset of X then we denote by $\langle A \rangle$ the fuzzy subset of X whose membership function $\mu_{\langle A \rangle}$ is equal to the characteristic function of A, that is,

$$\mu_{\langle A\rangle}(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in X \setminus A. \end{cases}$$

For brevity, we often refer to such fuzzy sets as crisp fuzzy sets.

For example, if $K_1, \dots, K_m$ are distinct subsets of I then the values of membership function $\beta_{\langle\{K_1,\dots,K_m\}\rangle}$ of the crisp subset $\langle\{K_1, \dots, K_m\}\rangle$ of $\mathcal{P}(I)$ are given by

$$\beta_{\langle\{K_1,\dots,K_m\}\rangle}(K) = \begin{cases} 1 & \text{if} \quad K \in \{K_1, \dots, K_m\} \\ 0 & \text{otherwise.} \end{cases}$$

This notation will be especially useful for one-element set of blocks $\{K\}$, $K \in \mathcal{P}(I)$ and fuzzy coalition $\langle\{K\}\rangle$ condensed into one "sure" block.

Every fuzzy coalition $\mathcal{L}$ is formed by a family of blocks $K_0, \dots, K_N$ each of which participates in $\mathcal{L}$ with a part of its total endeavour characterized by $\beta_{\mathcal{L}}(K_j)$, $j = 0, 1, \dots, N$. As each player $i \in I$ can participate in several blocks, it is useful to define the set of blocks $\mathcal{P}_i(I) = \{K \subset I : i \in K\}$ and then we may characterize his or her maximal activity in $\mathcal{L}$ by number $\tau_{\mathcal{L}}(i)$ where

$$\tau_{\mathcal{L}}(i) = \max(\beta_{\mathcal{L}}(K) : K \in \mathcal{P}_i(I)). \tag{2}$$

Let us note that in this way, each fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ determines a fuzzy subset $L_{\mathcal{L}}$ of I with membership function $\mu_{L_{\mathcal{L}}} = \tau_{\mathcal{L}}$.

Remark 1. It can easily be seen from (2) and the definition of $\langle\{K\}\rangle$ that, for each $K \in \mathcal{P}(I)$, we obtain $\mu_{L_{\langle\{K\}\rangle}}(i) \in \{0, 1\}$ for each $i \in I$.

As mentioned in the Introduction, the usual model (see, e. g., [1, 2, 3, 4]) defines a fuzzy coalition as a fuzzy subset L of I, $L \in \mathcal{F}(I)$, with membership function $\mu_L : I \to [0, 1]$ and it does not consider any blocks. The mutual correspondence between both approaches is worth rather more thorough analyzing. The first relation informing about it has already been given by Remark 1. In the first reading, (2) may look rather arbitrary, but we may see that it reflects more essential correspondence.

Remark 2. Let us consider a fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ and a block $K \in \mathcal{P}(I)$. Then K and $\mathcal{L}$ determine a fuzzy subset of I with membership function $\tau_K^{(\mathcal{L})} : I \to [0, 1]$ defined by

$$\begin{aligned} \tau_K^{(\mathcal{L})}(i) &= \beta_{\mathcal{L}}(K) \quad \text{for } i \in K \\ &= 0 \qquad\quad \text{else.} \end{aligned} \tag{3}$$

Lemma 1. Each fuzzy subset $L_{\mathcal{L}}$ of I determined by $\mathcal{L}$ through (2) is a fuzzy union of fuzzy blocks described by Remark 1; namely, for each $i \in I$,

$$\mu_{L_{\mathcal{L}}}(i) = \max\left(\tau_K^{(\mathcal{L})}(i) : K \in \mathcal{P}(I)\right).$$

P r o o f. The statement follows from (2) and from Remark 1. Namely,

$$\begin{aligned} & \max\left(\tau_K^{(\mathcal{L})}(i) : K \in \mathcal{P}(I)\right) = \\ = \; & \max\left[\max(\tau_K^{(\mathcal{L})}(i) : K \in \mathcal{P}_i(I)), \max(\tau_K^{(\mathcal{L})}(i) : K \in \mathcal{P}(I) \setminus \mathcal{P}_i(I))\right] = \\ = \; & \max\left[\max(\beta_{\mathcal{L}}(K) : K \in \mathcal{P}_i(I), 0)\right] = \\ = \; & \max\left(\beta_{\mathcal{L}}(K) : K \in \mathcal{P}_i(I)\right) = \mu_{L_{\mathcal{L}}}(i). \end{aligned}$$

□

After having introduced cooperative games with fuzzy coalitions defined as fuzzy sets of blocks, we should mention, at least briefly, the following formal parallel with the deterministic TU-games. Every fuzzy coalition $\mathcal{L}$ in our model can be represented by a vector of real numbers

$$(\beta_{\mathcal{L}}(K))_{K \in \mathcal{P}(I)} .$$

If we add one more assumption, namely, $\beta_{\mathcal{L}}(K_0) = 0$, which contradicts neither the above definition of fuzzy coalition, nor the intuitive idea of expected behaviour of players in fuzzy cooperative game (the empty coalition does not influence it, however admissible it could be), then the pair

$$(I, \beta_{\mathcal{L}})$$

fulfils all requirements of the definition of ordinary TU cooperative games. Though this way of consideration is not developed in this paper, it is useful to consider this fact as a source of eventual useful tools for further development of the suggested model of fuzzy cooperation.

Intermezzo 1: From Fuzzy Players to Fuzzy Blocks

In the previous section we have discussed how to generate a fuzzy subset of I from a fuzzy subset of $\mathcal{P}(I)$. The opposite direction was considered in [10, 11]. To recollect the main ideas of the procedure, we consider a fuzzy subset L of I, i. e., $L \in \mathcal{F}(I)$ with membership function τ_L.

Keeping the notation of crisp sets treated as special type of fuzzy sets (see the previous section), we may consider any crisp coalition, i. e., block, $K \in \mathcal{P}(I)$ for an extremal case of this sort of "fuzzy" coalition. To stress this interpretation of K, we denote by $\langle K \rangle \in \mathcal{F}(I)$ the fuzzy set with membership function $\tau_{\langle K \rangle}(i) = 1$ if $i \in K$ and $\tau_{\langle K \rangle}(i) = 0$ for $i \notin K$.

Then it is easy to see that for every $L \in \mathcal{F}(I)$ there exist $K_1, \ldots, K_k \in \mathcal{P}(I)$, such that $\{K_1, \ldots, K_k\} \subset \mathcal{P}(I)$, i. e., $K_j \neq K_\ell$ for $j \neq \ell$ and real numbers $b_{K_1}, b_{K_2}, \ldots, b_{K_k}$ such that

$$b_{K_j} > 0 \quad \text{for} \quad j = 1, \ldots, k, \quad b_{K_1} + \cdots + b_{K_k} = 1,$$

and

$$\tau_L(i) = b_{K_1} \cdot \tau_{\langle K_1 \rangle}(i) + \cdots + b_{K_k} \cdot \tau_{\langle K_k \rangle}(i) \quad \text{for all } i \in I.$$

We briefly denote the above relation among L, $\{K_1, \ldots, K_k\}$, $\{b_{K_1}, \ldots, b_{K_k}\}$ by

$$L = b_{K_1} \cdot K_1 + \cdots + b_{K_k} \cdot K_k$$

and call $\{K_1, \ldots, K_k\}$ *convex representation* of L with coefficients $\{b_{K_1}, b_{K_2}, \ldots, b_{K_k}\}$.

It is easy to verify that if $L = K$ for some block K, then there exists exactly one convex representation of L, namely, $L = b_{K_1} \cdot K_1$ with $K_1 = K$ and $b_{K_1} = 1$. On the other hand, if $L \in \mathcal{F}(I)$, is not a crisp fuzzy subset of I, then there often exist several convex representations of L.

The relation between fuzzy coalitions defined in $\mathcal{F}(I)$ and in $\mathcal{F}(\mathcal{P}(I))$ can be specified by means of the convex representations introduced here. Namely, it is easy to see that every convex representation $\{K_1, \ldots, K_k\}$ of a fuzzy coalition $L \in \mathcal{F}(I)$ with coefficients $\{b_{K_1}, \ldots, b_{K_k}\}$ defines a fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ formed by blocks $K_1, \ldots, K_k$ with membership function $\beta_{\mathcal{L}}$ derived by the following procedure.

Having $L \in \mathcal{F}(I)$ and some of its convex representations $\{K_1, \ldots, K_k\}$ with coefficients $b_{K_1}, \ldots, b_{K_k}$, it is easy to define fuzzy subset $\mathcal{L}$ of $\mathcal{P}(I)$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$ such that

$$\begin{aligned} \beta_{\mathcal{L}}(K) &= b_K \quad \text{if } K \in \{K_1, \ldots, K_k\} \\ &= 0 \quad\;\; \text{else.} \end{aligned}$$

On the other hand, it is also obvious that if $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with $\beta_{\mathcal{L}}$, and if

$$\sum\nolimits_{K \in \mathcal{P}(I)} \beta_{\mathcal{L}}(K) = 1$$

then the crisp blocks $K \in \mathcal{P}(I)$ for which $\beta_{\mathcal{L}}(K) > 0$, form a convex representation of some fuzzy coalition $L \in \mathcal{F}(I)$.

The procedures described in Lemma 1 and in the Intermezzo deserve a rather heuristic comment. It regards something what can be interpreted like certain sort of reversibility. On one hand, procedure (2) prescribes to every fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ a fuzzy set $L \in \mathcal{F}(I)$ (which is the fuzzy coalition in the traditional meaning). As the set $\mathcal{F}(\mathcal{P}(I))$ represents a much richer structure than $\mathcal{F}(I)$, it is natural that one $L \in \mathcal{F}(I)$ may correspond to several $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$. The correspondence is realized by means of (2). On the other hand, every fuzzy set from $\mathcal{F}(I)$ can be decomposed into a combination of crisp fuzzy coalitions $\{K_1, \ldots, K_m\}$ with coefficients b_{K_j}, $j = 1, \ldots, m$, and we may construct a fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ simply by putting $\beta_{\mathcal{L}}(K_j) = b_{K_j}$ for $j = 1, \ldots, m$. Note that these two procedures are not mutually inverse and, moreover, the letter does not cover the whole set $\mathcal{F}(\mathcal{P}(I))$. It means that there are fuzzy coalitions in $\mathcal{F}(\mathcal{P}(I))$ which cannot result from any convex combination of crisp fuzzy set.

Let us note, too, that a convex combination of crisp fuzzy coalitions $\{K_1, \ldots, K_k\}$ with $b_{K_j} = \beta_{\mathcal{L}}(K_j)$, $j = 1, \ldots, k$, may form a convex representation of some fuzzy subset from $\mathcal{F}(I)$ if and only if

$$\sum\nolimits_{j=1}^{k} b_{K_j} = \sum\nolimits_{j=1}^{k} \beta_{\mathcal{L}}(K_j) \leq 1,$$

where the case of strict inequality $<$ may be easily managed by proper and trivial manipulation with empty coalition. On the other hand, the validity of the opposite strict inequality $>$ in the above formula is connected with the situations in which the combination of crisp fuzzy coalitions cannot intermediate the relation between $\mathcal{F}(\mathcal{P}(I))$ and $\mathcal{F}(I)$.

3 Disjointness of Fuzzy Coalitions

Let us continue the analysis of fuzzy coalitions defined as fuzzy subsets of $\mathcal{P}(I)$.

For some purposes, it is useful to introduce the concept of *disjointness* of fuzzy coalitions. As they are fuzzy sets of blocks, it is natural to admit that their disjointness is a fuzzy relation over $\mathcal{F}(\mathcal{P}(I))$. It is represented by a fuzzy subset of $\mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I))$ with membership function $\delta : \mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I)) \to [0,1]$ such that for any pair of fuzzy coalitions $\mathcal{L}$, $\mathcal{M}$

(4) $$\delta(\mathcal{L},\mathcal{M}) = 1 - \max\left(\beta_{\mathcal{L}\cap\mathcal{M}}(K) : K \in \mathcal{P}(I)\right) = 1 - \max_{K\in\mathcal{P}(I)} \left(\min(\beta_{\mathcal{L}}(K),\, \beta_{\mathcal{M}}(K))\right).$$

Lemma 2. For any $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, $\delta(\mathcal{L},\mathcal{M}) = \delta(\mathcal{M},\mathcal{L})$, $\delta(\mathcal{L},\mathcal{L}) = 1 - \max(\beta_{\mathcal{L}}(K) : K \in \mathcal{P}(I))$. If $\emptyset \in \mathcal{F}(\mathcal{P}(I))$ is the empty fuzzy coalition ($\beta_\emptyset(K) = 0$ for all $K \in \mathcal{P}(I)$) then $\delta(\mathcal{L},\emptyset) = 1$.

Proof. The statements follow directly from (4). □

Remark 3. If K, $K' \in \mathcal{P}(I)$ are non-empty then $\delta(\langle K\rangle, \langle K'\rangle) = 0$ when $K = K'$, and $\delta(\langle K\rangle, \langle K'\rangle) = 1$ when $K \neq K'$.

Finally, it is useful to consider one concept which is important for the model of partnerships in blocks and fuzzy coalitions. After the period of negotiations among players and (which is much more significant) among blocks, some partnerships are admitted while others are rejected. The structure of cooperation which is finally realized, is called (in the deterministic as well as fuzzy TU-games) a *coalitional structure* and it is defined as a set of fuzzy coalitions $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m\}$ such that for each player $i \in I$

(5) $$\sum\nolimits_{j=1}^{m} \max\left\{\beta_{\mathcal{L}_j}(K) : i \in K\right\} = 1.$$

Lemma 3. For each coalitional structure $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m\}$ and each player $i \in I$,

$$\sum\nolimits_{j=1}^{m} \tau_{\mathcal{L}_j}(i) = 1.$$

Proof. The statement follows directly from (2) and (5). □

It is easy to verify also the validity of the following statements.

Remark 4. If $\{K_1, \ldots, K_m\}$ is a set of blocks, then $\{\langle K_1\rangle, \ldots, \langle K_m\rangle\}$ is a coalitional structure if and only if the blocks are pairwise disjoint and their union is equal to I. In particular, $\{\langle I\rangle\}$ is a coalitional structure.

Lemma 4. If $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ is a coalitional structure of fuzzy coalitions and $\mathcal{L}_0 = \langle\emptyset\rangle$, then $\{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_m\}$ is also a coalitional structure.

P r o o f. By Lemma 1, for $\mathcal{L}_0 = \langle\emptyset\rangle$, we have $\tau_{\mathcal{L}_0}(i) = 0$ for each $i \in I$. Since $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ is a coalitional structure, it follows from Lemma 3 that

$$\sum_{j=0}^{m} \tau_{\mathcal{L}_j}(i) = \sum_{j=1}^{m} \tau_{\mathcal{L}_j}(i) = 1$$

for each $i \in I$. Consequently, $\{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_m\}$ is a coalitional structure. □

Intermezzo 2: Fuzzy Quantities

Since the treatment of the coalitional characteristic function for TU-games with fuzzy coalitions (in the deterministic model it was denoted by v) demands certain elementary knowledge of fuzzy quantities, we recollect here briefly a few basic concepts.

Every *fuzzy quantity* a is defined as a fuzzy subset of R with membership function $\mu_a : R \to [0, 1]$. Here, we suppose that its support is a bounded subset of R.

The fuzzy quantities may be processed analogously to the ordinary numbers by means of so called *extension principle* (cf. [7, 9]). Namely, in our case we need to operate with their sums and their products with ordinary numbers. If $a,\, b \in \mathcal{F}(R)$ are fuzzy quantities and $r \in R$ is an ordinary real number, then the *sum* $a \oplus b$ is a fuzzy quantity with $\mu_{a\oplus b} : R \to [0, 1]$, where for each $x \in R$

$$\mu_{a\oplus b}(x) = \sup\left[\min(\mu_a(y),\, \mu_b(x - y))\right]. \tag{6}$$

Similarly, the *product* $r \cdot a$ is a fuzzy quantity with $\mu_{r\cdot a} : R \to [0, 1]$, where for each $x \in R$

$$\begin{aligned} \mu_{r\cdot a}(x) &= \mu_a(x/r) && && \text{for } r \neq 0, \\ \mu_{0\cdot a}(0) &= 1, & \mu_{0\cdot a}(x) &= 0 && \text{for } x \neq 0. \end{aligned} \tag{7}$$

Moreover, the models of rational behaviour and optimal decision-making frequently demand to compare numerical values – in our case the fuzzy quantities. There exist many approaches to the definition of ordering relation over fuzzy quantities (see, e. g. [9]). Here, we respect the methodological paradigm that ordering of vague elements is a vague relation, and define the *fuzzy ordering* $\succeq$ as a fuzzy relation over $\mathcal{F}(R)$. It is represented by a fuzzy subset of $\mathcal{F}(R) \times \mathcal{F}(R) \to [0, 1]$ whose membership function $\nu_{\succeq}$ is defined by

$$\nu_{\succeq}(a, b) = \sup\left[\min(\mu_a(x),\, \mu_b(y)) : x,\, y \in R,\, x \geq y\right]. \tag{8}$$

These concepts will be useful for analyzing the structure of pay-offs to fuzzy coalitions of blocks.

4 Characteristic Functions and Pay–offs of Fuzzy Coalitions

Similarly to other sections, we assume that it is given an ordinary TU-game (I, v). We have extended this classical model by admitting that there exist fuzzy coalitions as a specific form of vague cooperation of particular blocks of players. The fuzziness of cooperation among blocks naturally justifies the intuitive expectation that there would be some fuzzy features in the pay-offs of fuzzy coalitions, too. In this section, we extend the mapping v onto the set of fuzzy coalitions.

For every fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$ we first define the real number

$$b_{\mathcal{L}}(K) = \beta_{\mathcal{L}}(K) \cdot v(K).$$

Since we assume that $v(K)$ is nonnegative, we know that $b_{\mathcal{L}}(K) \leq v(K)$. Now we define a fuzzy quantity $w(\mathcal{L}) \in \mathcal{F}(R)$ with membership function $\mu_{\mathcal{L}} : R \to [0,1]$ such that for each $x \in R$

$$\begin{aligned} \mu_{\mathcal{L}}(x) &= \max\left(\beta_{\mathcal{L}}(K) : K \in \mathcal{P}(I),\, b_{\mathcal{L}}(K) = x\right), \\ &= 0 \qquad \text{if } b_{\mathcal{L}}(K) \neq x \text{ for all } K \in \mathcal{P}(I). \end{aligned} \tag{9}$$

The fuzzy quantities $w(\mathcal{L})$ for $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ form *fuzzy characteristic function* of the TU-game (I, v) with fuzzy coalitions. The pair (I, w) characterizes the cooperation in the considered game. In accordance with [8], it is correct to consider the pair (I, w) for a TU-game with fuzzy coalitions and fuzzy pay-offs extending (and generalizing) the original deterministic game (I, v).

Lemma 5. If $K \in \mathcal{P}(I)$ then, for $\langle K \rangle$ as defined in Section 2, $\mu_{\langle K \rangle}(v(K)) = 1$ and $\mu_{\langle K \rangle}(x) = 0$ for $x \neq v(K)$.

P r o o f. Let $K \in \mathcal{P}(I)$, $\langle K \rangle \in \mathcal{F}(\mathcal{P}(I))$. Then

$$\begin{aligned} \mu_{\langle K \rangle}(x) &= \beta_{\langle K \rangle}(K) = 1 \quad \text{for} \quad x = b_{\langle K \rangle}(K) = 1 \cdot v(K) = v(K) \\ &= 0 \qquad\qquad\qquad \text{for} \quad x \neq v(K), \end{aligned}$$

as follows from (9) and from definition of $\langle K \rangle$ (see Section 2). □

Remark 5. Since $v(K) \geq 0$ for all $K \in \mathcal{P}(I)$, it is evident that, for all $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$, we have $\mu_{\mathcal{L}}(x) = 0$ if $x < 0$. Moreover, for the empty block K_0 and the fuzzy coalition $\langle K_0 \rangle$, we have $\mu_{\langle K_0 \rangle}(0) = 1$, as follows from Lemma 5, equality (9) and the definition of v in Section 2.

Lemma 6. For the empty fuzzy coalition $\emptyset$, $w(\emptyset)$ is the empty fuzzy subset of R, i. e., $\mu_{\emptyset}(x) = 0$ for all $x \in R$.

P r o o f. The statement follows from (9) and from Remark 5: If $\emptyset$ is the empty fuzzy coalition then $\beta_{\emptyset}(K) = 0$ for all $K \in \mathcal{P}(I)$. Due to (9)

$$\mu_{\emptyset}(x) = 0 \quad \text{for all } x \in R,$$

which means that $w(\emptyset)$ is the empty fuzzy quantity. □

It is useful to comment the above definition and its interpretation. In our model, coalitions are the cooperative groups of blocks with some secondary consequences for particular players. Members of each block have homogeneous interests and the coalitions are formed to coordinate the activities of blocks. From this point of view the component of the compromise in the coalitions' behaviour is rather marginalized and the main emphasis is focused on the choice of the block (among those which form the coalition) whose profit appears to be most acceptable for the whole coalition and may be, eventually, distributed among all its members. In this sense, formula (9) and related concepts well reflect the monotonous (not necessarily additive) evaluation of the outcome of coalitions.

5 Monotonicity of Fuzzy Characteristic Function

Some of the most significant properties of the characteristic functions of deterministic TU-games are connected with some forms of their additivity (especially with superadditivity). The methodology of fuzzy set theory used in (9), as well as its heuristics mentioned in the previous paragraph, gives emphasis rather to the monotonicity of the fuzzy characteristic function.

Theorem 1. The fuzzy characteristic function $w : \mathcal{F}(\mathcal{P}(I)) \to \mathcal{F}(R)$ is monotonous in the sense that if $\mathcal{L}, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, $\beta_{\mathcal{L}}(K) \geq \beta_{\mathcal{M}}(K)$ for all $K \in \mathcal{P}(I)$, then $\nu_{\succeq}(w(\mathcal{L}), w(\mathcal{M})) \geq \nu_{\succeq}(w(\mathcal{M}), w(\mathcal{L}))$. Moreover, for each $x \in R$ there exists $y \in R$ such that $y \geq x$ and $\mu_{\mathcal{L}}(y) \geq \mu_{\mathcal{M}}(x)$.

P r o o f. The theorem follows from (9) almost immediately. For every $K \in \mathcal{P}(I)$, $\beta_{\mathcal{L}}(K) \geq \beta_{\mathcal{M}}(K)$ and this, together with (9) and (8), immediately implies both statements. □

The following two corollaries are immediate consequences of the theorem.

Corollary 1. For every pair of fuzzy coalitions $\mathcal{L}, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, the following relation holds

$$\nu_{\succeq}\left(w(\mathcal{L} \cup \mathcal{M}),\, w(\mathcal{L}) \cup w(\mathcal{M})\right) \geq \nu_{\succeq}\left(w(\mathcal{L}) \cup w(\mathcal{M}),\, w(\mathcal{L} \cup \mathcal{M})\right),$$

where $w(\mathcal{L}) \cup w(\mathcal{M}) \in \mathcal{F}(R)$ is the usual union of fuzzy quantities $w(\mathcal{L})$ and $w(\mathcal{M})$.

Corollary 2. For every pair of fuzzy coalitions $\mathcal{L}, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, the following relation holds

$$\nu_{\succeq}\left(w(\mathcal{L}),\, w(\mathcal{L}) \cap w(\mathcal{M})\right) \geq \nu_{\succeq}\left(w(\mathcal{L}) \cap w(\mathcal{M}),\, w(\mathcal{L})\right),$$

where $w(\mathcal{L}) \cap w(\mathcal{M}) \in \mathcal{F}(R)$ is the usual intersection of fuzzy quantities $w(\mathcal{L})$ and $w(\mathcal{M})$.

Theorem 2. If, in the original deterministic game (I, v), for each $K \subset I$, $v(I) \geq v(K)$, and if a fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ includes the block I with full intensity, that is, $\beta_{\mathcal{L}}(I) = 1$, then $\nu_{\succeq}(w(\mathcal{L}), w(\mathcal{M})) = 1$ for each $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$.

P r o o f. The statement follows from (9) and Theorem 1. If $v(I) \geq v(K)$ for all $K \subset I$ and $\beta_{\mathcal{L}}(I) = 1$ then $b_{\mathcal{L}}(I) \geq b_{\mathcal{L}}(K)$ for all $K \in \mathcal{P}(I)$. Due to (9), $\mu_{\mathcal{L}}(v(I)) = 1$ and for any $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ and any $K' \in \mathcal{P}(I)$, $\mu_{\mathcal{M}}(v(K')) > 0$ means that $v(K') \leq v(I)$. It means that the statement follows from (8). □

As an immediate consequence, we have the following corollary.

Corollary 3. If the original TU-game (I, v) is superadditive in the usual sense (that is, $v(K \cup K') \geq v(K) + v(K')$ for disjoint K, K'), and if $v(K) \geq 0$ for all $K \subset I$ then $\nu_{\succeq}(w(\langle I \rangle),\, w(\mathcal{L})) = 1$ for each $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$.

6 Additivity of Fuzzy Pay–offs

In the previous sections, we have stressed the point of view due to which the fundamental relations among fuzzy coalitions and blocks of players are based on the monotonicity principle, briefly mentioned in the previous section. Nevertheless, the coalitional pay-offs represented by fuzzy characteristic functions, are of quantitative character and their distribution to players can be, at least partly, described by algebraical tools. In certain sense the model resembles the model formulated in [8]. In fact, the essential gap between ordinary coalitions of players in [8] and fuzzy coalitions of blocks in this papers significantly changes the properties of the model and results from [8] cannot be mechanically transmitted to the above concepts. Nevertheless, it is appropriate to introduce the following ideas.

Let us consider, throughout this section, a TU-game (I, v) with fuzzy coalitions from $\mathcal{F}(\mathcal{P}(I))$ and with fuzzy characteristic function $w : \mathcal{F}(\mathcal{P}(I)) \to \mathcal{F}(R)$ whose values $w(\mathcal{L})$ are fuzzy quantities.

Analogously to some other cases considered in this paper, we accept the methodological paradigm that the properties of vague phenomena like fuzzy coalitions or fuzzy characteristic functions are to be vague (i. e., fuzzy), as well. In this section, we deal with convexity and superadditivity as fuzzy properties of TU-games with fuzzy coalitions. It means that they are identical with some fuzzy subsets of the set of such games – let us denote the set of all TU-games with fuzzy coalitions over the set of players I by $\Gamma(I)$.

The *fuzzy convexity* of games is defined as a fuzzy subset of $\Gamma(I)$ with characteristic function $\kappa : \Gamma(I) \to [0, 1]$ such that for each (I, v) the value $\kappa(v)$ determines the possibility that (I, v) is convex, which is defined as follows:

$$(10) \quad \kappa(v) = \min\left(\nu_{\succeq}(\widehat{w}(\mathcal{L} \cup \mathcal{M}) \oplus \widehat{w}(\mathcal{L} \cap \mathcal{M}),\, \widehat{w}(\mathcal{L}) \oplus \widehat{w}(\mathcal{M})) : \mathcal{L},\, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I))\right),$$

where $\widehat{w}$ are fuzzy quantities with normalized membership function $\widehat{\mu}_{\mathcal{L}} : R \to [0, 1]$,

$$(11) \qquad \widehat{\mu}_{\mathcal{L}}(x) = \mu_{\mathcal{L}}(x) \,/(\sup(\mu_{\mathcal{L}}(y) : y \in R))$$

for all $x \in R$, where $\mu_{\mathcal{L}}$ is defined by (9), (this normalization means that the values of $\nu_{\succeq}(\cdot, \cdot)$ in (10) are equal to 1 if $w(\mathcal{L} \cup \mathcal{M}) \oplus w(\mathcal{L} \cap \mathcal{M}) \succeq w(\mathcal{L}) + w(\mathcal{M})$ as follows from (8)). Of course, if $\mu_{\mathcal{L}}(x) = 0$ for all $x \in R$ then we put $\widehat{\mu}_{\mathcal{L}}(x) = 0$ for all $x \in R$, as well. The above notation deserves a comment. The values $\kappa(v)$ are related to formally deterministic games (I, v). We have to keep in mind that (I, v) generates also the fuzzy function w and, consequently, $\kappa(v)$ can be defined by means of w.

This can be easily interpreted as the possibility that

$$w(\mathcal{L} \cup \mathcal{M}) \oplus w(\mathcal{L} \cap \mathcal{M}) \succeq w(\mathcal{L}) \oplus w(\mathcal{M})$$

for all pairs of fuzzy coalitions $\mathcal{L}$, $\mathcal{M}$ with respect to the normalization of fuzzy characteristic function.

The concept of fuzzy superadditivity of TU-games appears to be rather more complicated as it demands the disjointness of the pairs of relevant coalitions. Let us recollect that it itself is a fuzzy relation between fuzzy coalitions (see Section 2, formula (4)) described as a fuzzy subset of $\mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I))$ with membership function $\delta(\cdot,\cdot)$ (see (4)). Then the *fuzzy superadditivity*, analogously to fuzzy convexity, is a fuzzy property of TU-games with fuzzy coalitions, and it is described by a fuzzy subset of $\Gamma(I)$. In this case, its membership function $\sigma : \Gamma(I) \to [0,1]$ is defined for every (I, v) as follows

$$\sigma(v) = \max\left(1 - \delta(\mathcal{L},\mathcal{M}),\, \nu_{\succeq}(\widehat{w}(\mathcal{L} \cup \mathcal{M}),\, \widehat{w}(\mathcal{L}) \oplus \widehat{w}(\mathcal{M})) : \mathcal{L},\, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I))\right), \tag{12}$$

where, analogously to the previous case of convexity, $\widehat{w}$ are fuzzy quantities derived from w by normalization of the membership function (11). Even their motivation is analogous.

This, seemingly rather complicated, definition means that each game belongs to the fuzzy set of superadditive games the more the higher is the possibility that

$$w(\mathcal{L} \cup \mathcal{M}) \succeq w(\mathcal{L}) \oplus w(\mathcal{M})$$

for coalitions which are disjoint enough.

Remark 6. Evidently, if we limit our attention to the fuzzy coalitions with at least one block with full participation, i. e., such $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ for which $\beta_{\mathcal{L}}(K) = 1$ for at least one $K \in \mathcal{P}(I)$, then the definitional formulas (10), (11) may be used with values $w(\cdot)$ instead of $\widehat{w}(\cdot)$, as follows from (8) and (9).

Let us note that the above notions of fuzzy convexity and fuzzy superadditivity for fuzzy coalitions formed by blocks essentially differ from similarly named properties defined for (crisp!) coalitions either in deterministic games with crisp pay-offs (cf., for example [5, 12]) or in fuzzy games with fuzzy pay-offs (see [8]). Relation between these approaches, if any, is evidently complex and deserves intensive investigation.

Theorem 3. Let us consider a TU-game (I, v) with fuzzy coalitions such that for any $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, $\mathcal{L} \neq \mathcal{M}$, and any $K \in \mathcal{P}(I)$ if $\beta_{\mathcal{L}}(K) > 0$ then $\beta_{\mathcal{M}}(K) = 0$. Then

$$\kappa(v) = \sigma(v).$$

Proof. The statement follows from (11) and (12). Under the assumptions of the theorem, any pair of different fuzzy coalitions $\mathcal{L}$, $\mathcal{M}$ is fuzzy disjoint, i. e. $\delta(\mathcal{L},\mathcal{M}) = 0$ and $\mathcal{L} \cap \mathcal{M}$ is empty. Using Lemma 6 it is easy to see that (10) immediately implies (12). □

7 Fuzzy Core

The concept of core belongs to the basic solution concepts of the classical TU-games and it is useful to mention, at least briefly, its analogy for our type of games with fuzzy coalitions. Let us note that, similarly to the previous concepts of convexity and superadditivity, a concept of core of the games with fuzzy characteristic functions (but crisp coalitions) was considered in [8], too. On the other hand, the core of TU-games with fuzzy coalitions from $\mathcal{F}(I)$ and crisp pay-offs is investigated in numerous works, e. g., in [1]. The admissibility of fuzzy coalitions does change the model so significantly that any mechanical transmission of the results given in [8] or [1] to our type of game is practically impossible.

Also in the case of core we preserve the methodological paradigm that concepts related to fuzzified TU-games are to be fuzzy, too. It follows that the core of TU-game with fuzzy coalitions (and their fuzzy pay-offs) is a fuzzy subset of R^n representing the distribution of the total pay-off among the individual players. This distribution is expected to respect the participation of the players in blocks and participation of blocks in fuzzy coalitions which is expressed by the fuzzy characteristic function w.

As mentioned, the core is defined as a fuzzy subset C of R^n. Its membership function $\gamma_C : R^n \to [0,1]$ is constructed as follows. For each $r \in R$, let us denote by $\langle r \rangle$ the fuzzy quantity condensed into a single value r, that is,

$$\mu_{\langle r \rangle}(r) = 1, \quad \mu_{\langle r \rangle}(x) = 0 \quad \text{for } x \neq r. \tag{13}$$

Hence, also $\langle \sum_{i \in I} x_i \rangle$ and $\langle \sum_{i \in K} x_i \rangle$ are fuzzy quantities with single possible value. Let us, further, denote for any $\boldsymbol{x} \in R^n$ and any $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ the vector $\boldsymbol{x}^{(\mathcal{L})} = (x_i^{(\mathcal{L})})_{i \in I} \in R^n$, where

$$x_i^{(\mathcal{L})} = x_i \cdot \tau_{\mathcal{L}}(i) \quad \text{for each } i \in I$$

and where values $\tau_{\mathcal{L}}(i)$ are defined by (2) in Section 2. Then we may denote

$$\pi(\boldsymbol{x}) = \min \left(\nu_{\succeq} \left(\left\langle \sum_{i \in I} x(\mathcal{L})_i \right\rangle, \widehat{w}(\mathcal{L}) \right) : \mathcal{L} \in \mathcal{F}(\mathcal{P}(I)) \right), \tag{14}$$

where $\widehat{w}(\mathcal{L})$ are the normalized fuzzy quantities defined by (11), and put

$$\begin{aligned} \gamma_C(\boldsymbol{x}) &= \pi(\boldsymbol{x}) \quad \text{if } \textstyle\sum_{i \in I} x_i \leq v(I) \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{15}$$

for the membership function of *fuzzy core* C.

Lemma 7. For any $\boldsymbol{x} \in R^n$,

$$\gamma_C(\boldsymbol{x}) = \min \left(\pi(\boldsymbol{x}), \nu_{\succeq} \left(w(\langle I \rangle), \left\langle \sum_{i \in I} x_i \right\rangle \right) \right).$$

P r o o f. For the fuzzy coalition $\langle I \rangle$ with single possible block I and $\beta_{\langle I \rangle}(I) = 1$, the fuzzy quantity $w(\langle I \rangle)$ is condensed in a single possible value $v(I)$, i. e., $\mu_{\langle I \rangle}(v(I)) = 1$, $\mu_{\langle I \rangle}(r) = 0$ for $r \neq v(I)$ (cf. (12) and Lemma 5). It follows that we may include the condition

$$\sum_{i \in I} x_i \leq v(I)$$

into the definitional formula (15) as the demand of fulfilling

$$w(\langle I\rangle) \succeq \left\langle \sum_{i\in I} x_i \right\rangle .$$

This implies the validity of the statement, because for $\langle I\rangle$, we have $\boldsymbol{x}^{(\langle I\rangle)} = \boldsymbol{x}$ for each $\boldsymbol{x} \in R^n$. □

The above definition of fuzzy core rather copies the usual deterministic one (cf. [5, 12], e. g.) and the one related to crisp coalitions with fuzzy pay-offs (see [8], e. g.) in the sense that it closely relates the core to the crisp coalition of all players. In this model, stressing the fuzziness of coalitions, this appears to be rather limiting, and a generalization may be desirable. The following one connects the concept of core with the coalitional structure (5), which approach can also be found in some papers on the deterministic TU-games, see [8].

Respecting the above notations, we define the *generalized fuzzy core* as a fuzzy subset C^* of R^n with membership function $\gamma_C^* : R^n \to [0,1]$ defined as follows. For each coalitional structure $\boldsymbol{\mathcal{L}} = \{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ (see (5) and Lemma 3) we denote

$$w(\boldsymbol{\mathcal{L}}) = w(\mathcal{L}_1) \oplus \cdots \oplus w(\mathcal{L}_m),$$

and, for each $\boldsymbol{x} \in R$, we introduce $\rho(\boldsymbol{x})$ by

(16) $$\rho(\boldsymbol{x}) \;=\; \max\Big[\min\Big(\nu_{\succeq}\Big(\widehat{w}(L_j), \Big\langle \sum_{i\in I} x_i^{(\mathcal{L}_j)}\Big\rangle\Big) : \mathcal{L}_j \in \boldsymbol{\mathcal{L}}\Big) : \boldsymbol{\mathcal{L}} \subset \mathcal{F}(\mathcal{P}(I)),\ \boldsymbol{\mathcal{L}} \text{ is a coalitional structure.}\Big]$$

Then we define γ_C^* by

(17) $$\gamma_C^* = \min(\rho(\boldsymbol{x}),\ \pi(\boldsymbol{x})).$$

Lemma 8. If the original TU-game (I, v) is superadditive in the sense that

$$v(K \cup K') \geq v(K) + v(K')$$

for disjoint K, $K' \in \mathcal{P}(I)$, and if (I, w) is the TU-game with fuzzy coalitions extending (I, v) due to (9) then the set $\{\langle I\rangle\}$ is the coalitional structure fulfilling the demands of definition (16).

P r o o f. It is obvious that the one-element set of fuzzy coalitions $\{\langle I\rangle\}$ is a coalitional structure. Moreover, $\mu_{\langle I\rangle}(v(I)) = 1$ and $\mu_{\langle I\rangle}(x) = 0$ for $x \neq v(I)$, as follows from Lemma 7. Due to Theorem 2 and Corollary 3, if for some $\boldsymbol{x} \in R^n$

$$\sum_{i\in I} x_i \leq v(I)$$

then

$$\nu_{\succeq}\Big(w(\langle I\rangle), \sum_{i\in I} x_i\Big) = 1$$

and $\rho(\boldsymbol{x}) = 1$. □

Theorem 4. If the original deterministic game is superadditive in the usual sense (cf. Observation 14) then the generalized fuzzy core is identical with the fuzzy core (15), i. e., $\gamma_C(\boldsymbol{x}) = \gamma^*_C(\boldsymbol{x})$ for all $\boldsymbol{x} \in R^n$.

P r o o f. The theorem follows immediately from Lemma 8 and Lemma 7. □

8 Modal Blocks

The concept of fuzzy coalition defined as fuzzy set of crisp blocks does not demand any further properties which are to be fulfilled by the membership function $\beta_{\mathcal{L}}$. It gives a significant freedom in the construction of particular models of cooperation. On the other hand, it leads in some cases to rather complex formal constructions which are useful only for the precise introduction of intuitively lucid objects and properties. Such construction is, for example, the normalization of the fuzzy characteristic function w by means of (11) and $\widehat{w}$, introduced in Section 6 and used also in Section 7.

The definitions and following results using the normalization can be simplified if we slightly modify the concept of fuzzy coalition. Namely, if we demand that every fuzzy coalition $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0, 1]$ has to contain at least one *modal block*, i. e., such $K \in \mathcal{P}(I)$ for which $\beta_{\mathcal{L}}(K) = 1$ and $K \neq \emptyset$.

This idea was marginally admitted in Remark 6. Here, we discuss its meaning and consequences, at least a little more thoroughly. Such modified model of fuzzy coalition brings not only an alternative formal representation of vague cooperation, but it demands also a specific interpretation.

If we restrict the concept of fuzzy coalition introduced in Section 2 to *modal fuzzy coalitions* defined as fuzzy subsets of $\mathcal{P}(I)$ containing at least one modal block, we accept also a modified interpretation of coalitional cooperation. Let us recall that in the preceding sections, we have turned away from the idea of fuzzy coalition as fuzzy set of players to the notion of fuzzy set of blocks of identically motivated players. The modification presented in this section reflects the hidden presumption that every fuzzy coalition originates as an initiative of (at least) one block which devotes all its power to the activities and interests represented by that coalition. Other blocks may, more or less, participate in those activities by some part of their endeavour. The definitive structure of the fuzzy coalition of the considered type follows from the superposition of all activities of blocks forming it. This interpretation also well justifies some previous definitions (e. g., (2)). If a player is a member of several blocks participating in a fuzzy coalition then he includes one activity – the maximum of the activities of the blocks which contribute to the coalition.

The concept of modal fuzzy coalitions simplifies some further concepts and properties derived in the previous sections for general fuzzy coalitions. Let us start with some elementary consequences of the modality of $\beta_{\mathcal{L}}$.

In the remaining part of this section, we denote by $\widehat{\mathcal{F}}(\mathcal{P}(I))$, $\widehat{\mathcal{F}}(\mathcal{P}(I)) \subset \mathcal{F}(\mathcal{P}(I))$, the set of all modal fuzzy coalitions.

Remark 7. If $\mathcal{L} \in \widehat{\mathcal{F}}(\mathcal{P}(I))$ is a fuzzy coalition with membership function $\beta_{\mathcal{L}}$ and with modal block $K \in \mathcal{P}(I)$, $\beta_{\mathcal{L}}(K) = 1$, then the membership function $\tau_{\mathcal{L}} : I \to [0, 1]$ defined by (2) has also some modal elements, because $\tau_{\mathcal{L}}(i) = 1$ for all $i \in K$.

Remark 8. If $\mathcal{L}$, $\mathcal{M} \in \widehat{\mathcal{F}}(\mathcal{P}(I))$ are modal fuzzy coalitions, then evidently $\mathcal{L} \cup \mathcal{M}$ is a modal fuzzy coalition. However $\mathcal{L} \cap \mathcal{M}$ need not be modal.

Remark 9. If $K \in \mathcal{P}(I)$ then $\langle K \rangle$ is evidently a modal fuzzy coalition.

One of the properties formulated in Lemma 2 can be essentially simplified.

Remark 10. If $\mathcal{L} \in \widehat{\mathcal{F}}(\mathcal{P}(I))$ is a modal fuzzy coalition and $\delta : \mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I)) \to [0, 1]$ is defined by (4) then $\delta(\mathcal{L}, \mathcal{L}) = 0$.

Even the elementary properties of coalitional structures of modal fuzzy coalitions become somewhat more lucid.

Remark 11. If $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m\}$ is a coalitional structure of modal fuzzy coalitions and if $K_j \in \mathcal{P}(I)$ is a modal block of $\mathcal{L}_j$, then $\beta_{\mathcal{L}_k}(K_j) = 0$ for all $\mathcal{L}_k$, $k \neq j$, $k \in \{1, \ldots, m\}$.

The modality assumption offers more significant consequences for the fuzzy pay-offs.

Lemma 9. Let $\mathcal{L} \in \widehat{\mathcal{F}}(\mathcal{P}(I))$ be modal fuzzy coalition and $K \in \mathcal{P}(I)$ be its modal fuzzy block. Let $\mu_{\mathcal{L}}$ be the membership function of $w(\mathcal{L})$ defined by (9). Then $\mu_{\mathcal{L}}(v(K)) = 1$.

Proof. If K is a modal block of L then $\beta_{\mathcal{L}}(K) = 1$ and $b_{\mathcal{L}}(K) = v(K)$. Due to (9), for $x = v(K)$,

$$\mu_{\mathcal{L}}(x) = \max\left(\beta_{\mathcal{L}}(K') : K' \in \mathcal{P}(I),\, x \in K'\right) = \beta_{\mathcal{L}}(K) = 1. \qquad \square$$

The previous result enables us to simplify some further concepts. The monotonicity (cf., Theorem 1) turns into the following relation.

Theorem 5. The fuzzy characteristic function $w : \mathcal{F}(\mathcal{P}(I)) \to \mathcal{F}(R)$ fulfils for modal fuzzy coalitions $\mathcal{L}$, $\mathcal{M} \in \widehat{\mathcal{F}}(\mathcal{P}(I))$ such that $\beta_{\mathcal{L}}(K) \geq \beta_{\mathcal{M}}(K)$ for all $K \in \mathcal{P}(I)$ the relation

$$\nu_{\succeq}(w(\mathcal{L}),\, w(\mathcal{M})) = 1.$$

Proof. Due to Theorem 1,

$$\nu_{\succeq}(w(\mathcal{L}),\, w(\mathcal{M})) \geq \nu_{\succeq}(w(\mathcal{M}),\, w(\mathcal{L})).$$

According to definition (8), we have

$$\nu_{\succeq}(w(\mathcal{M}),\, w(\mathcal{L})) = \sup\left[\min\left(\mu_{\mathcal{M}}(x),\, \mu_{\mathcal{L}}(y)\right) : x,\, y \in R,\, x \geq y\right].$$

For a modal block K of $\mathcal{M}$, $\mu_{\mathcal{M}}(v(K)) = 1$ as follows from Lemma 9. Due to the assumptions, K is also a modal block for $\mathcal{L}$. Consequently, $\mu_{\mathcal{L}}(v(K)) = 1$, as well. It follows that $\min(\mu_{\mathcal{L}}(v(K)),\, \mu_{\mathcal{M}}(v(K))) = 1$ and, consequently,

$$\nu_{\succeq}(w(\mathcal{M}),\, w(\mathcal{L})) = 1$$

which proves the statement. □

Lemma 10. Let $\mathcal{L} \in \widehat{\mathcal{F}}(\mathcal{P}(I))$ be modal fuzzy coalition and let $\widehat{w} : \mathcal{F}(\mathcal{P}(I)) \to \mathcal{F}(R)$ be the fuzzy function defined by (11). Then $\widehat{w}(\mathcal{L}) = w(\mathcal{L})$.

P r o o f. The statement follows directly from (11) and Lemma 9. □

The next statement is an immediate consequence of Lemma 10. Let us recollect Remark 7, too.

Theorem 6. If we admit only the modal fuzzy coalitions from $\hat{\mathcal{F}}(\mathcal{P}(I))$ then, for each $(I, v) \in \Gamma(I)$,

$$\begin{aligned}
\kappa(v) &= \min\left(\nu_{\succeq}(w(\mathcal{L}\cup\mathcal{M}) \oplus w(\mathcal{L}\cap\mathcal{M}),\, w(\mathcal{L}) \oplus w(\mathcal{M})) : \mathcal{L},\, \mathcal{M} \in \widehat{\mathcal{F}}(\mathcal{P}(I))\right),\\
\sigma(v) &= \max\left(1 - \delta(\mathcal{L},\mathcal{M}),\, \nu_{\succeq}(w(\mathcal{L}\cup\mathcal{M}),\, w(\mathcal{L}) \oplus w(\mathcal{M})) : \mathcal{L},\, \mathcal{M} \in \widehat{\mathcal{F}}(\mathcal{P}(I))\right),\\
\pi(\boldsymbol{x}) &= \min\left(\nu_{\succeq}\left(\left\langle \sum_{i\in I} x_i^{(\mathcal{L})}\right\rangle, w(\mathcal{L})\right) : \mathcal{L} \in \widehat{\mathcal{F}}(\mathcal{P}(I))\right),
\end{aligned}$$

and

$$\begin{aligned}
\rho(\boldsymbol{x}) = \max\Big[&\min\left(\nu_{\succeq}\left(w(\mathcal{L}_j), \left\langle \sum_{i\in I} x_i^{(\mathcal{L}_j)}\right\rangle\right) : \mathcal{L}_j \in \boldsymbol{\mathcal{L}}\right)\\
&\boldsymbol{\mathcal{L}} \subset \widehat{\mathcal{F}}(\mathcal{P}(I)),\ \boldsymbol{\mathcal{L}} \text{ is a coalitional structure}\Big].
\end{aligned}$$

9 Remarks

The behaviour of players in TU-games with vague cooperation in which a player distributes his endeavour among several cooperating structures is natural and usual but, on the other side, also rather strange and contradictory. Each player has essentially exactly one interest – his own profit. But he is able to part his activities going to its achievement into several "loyalities" to several group of close partners. This partition of cooperative activities includes some respect to the preferences of coalitional partners in order to coordinate the activities targeting to the optimal results. It appears that this multipolarity of interests is one of the essential sources of vagueness existing in the attitudes of players.

The presented model suggests formal tools for an adequate treatment of such situations. The players, however they have their individual preferences and earn individual pay-offs, act as members of compact blocks, maybe in several of them, parallelly, and these blocks are the real acting agents of (fuzzy) coalitions. This model probably reflects the realistic situation of the game and it is lucid enough to be manageable with the fuzzy set theoretical tools. The elementary properties of its components presented in this paper deserve some brief comment.

First of all, let us emphasize that in spite of the marginal similarity with some models of fuzzy TU-games (like [1, 2] or [11]) the presented model is essentially different, and we do not see how to transfer concepts and results of those previous models without their thorough modification.

It is also useful to point out that the presented model combines fuzzy coalitions (known from [1, 2, 3]) and fuzzy characteristic function (investigated in [8]) into a unified and consistent model in which the combination of both types of fuzziness follows from the relations in the modelled reality.

Finally, let us note that the above formulation of the principles of the model is quite general and wide. It can be simplified without essential limitation of its positive features if we in its next analysis accept a quite rational assumption formulated in Section 8. The modality assumption guarantees that, for each pair of fuzzy quantities a, b, at least one of the values $\nu_{\succeq}(a, b)$ and $\nu_{\succeq}(b, a)$ is equal to 1, which in our model significantly simplifies all definitions and results in which the values of $w(\mathcal{L})$ are compared.

It can be also inspirative to reconsider the definition of the fuzzy core of our model of game. Namely, it is possible to analyze the concept of solution in which the total profit $w(\langle I \rangle)$ is distributed not among individual players by means of some imputations $\boldsymbol{x} = (x_i)_{i \in I}$ but among blocks. Such solution concept, if admitted, essentially changes the philosophy of the TU-games and the basic paradigm of their individual character, and in this sense its eventual acceptance demands a deeper analysis.

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