# Finding Solution of Coalition Games by Bargainining Schemes


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# Coalition Games

(von Neumann, Morgenstern; 1953)

- models of interacting decision-makers that focus on the behavior of groups of players
- every coalition acts as an collective decision maker in the name of its members

A coalition game is specified by

- a set of players
- a set of coalitions
- a payoff of every coalition

A solution of a game is a predicted set of payoffs distributed among players.

# Games with Fuzzy Coalitions

(J.-P. Aubin, 1974)

| | |
|---|---|
| $N = \{1, \ldots, n\}$ | set of players |
| $a = (a_1, \ldots, a_n) \in [0, 1]^n$ | fuzzy coalition |
| $a \in \{0, 1\}^n$ | crisp coalition (subset of $N$) |

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**Definition**

*A game (with fuzzy coalitions) is a function*

$$v : [0,1]^n \to \mathbb{R} \quad \textit{with } v(0) = 0.$$

# Core of Game

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| $x \in \mathbb{R}^n$ | vector of individual payoffs |
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Definition (Aubin; 1974)

*Let $v$ be a game. The core of $v$ is the set*

$$\mathbf{C}(v) = \left\{x \in \mathbb{R}^n \mid \langle 1, x \rangle = v(1) \text{ and } \langle a, x \rangle \geq v(a),\ \forall a \in [0,1]^n \setminus \{1\}\right\}$$

Put

$$C_a(v) = \begin{cases} \{x \in \mathbb{R}^n \mid \langle a, x \rangle \geq v(a)\}, & \text{if } a \in [0,1]^n \setminus \{1\}, \\ \{x \in \mathbb{R}^n \mid \langle 1, x \rangle = v(1)\}, & \text{if } a = 1. \end{cases}$$

Then

$$\mathbf{C}(v) = \bigcap_{a \in [0,1]^n} C_a(v)$$

# Characterizations of Core

Theorem (Aubin, 1981)

*Let $v$ be a PH and superadditive game. If $v$ is continuously differentiable at 1, then $\mathbf{C}(v) \neq \emptyset$ and*

$$\mathbf{C}(v) = \{\nabla v(1)\}.$$

Theorem (Tijs et al., 2003)

*Let $v$ be a game such that for every $a, b, d, b + d \in [0, 1]^n$:*

$$a \leq b \quad \Rightarrow \quad v(a + d) - v(a) \leq v(b + d) - v(b).$$

*Then $\mathbf{C}(v) \neq \emptyset$ and*

$$\mathbf{C}(v) = \bigcap_{a \in \{0,1\}^n} C_a(v).$$

# Examples of Cores

$N = \{1, 2\}$

Example (empty core)

$$u(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ 1, & \textit{otherwise.} \end{cases}$$

$\mathbf{C}(u) = \emptyset$ *since the hyperplane* $x_1 + x_2 = 1$ *misses* $\frac{2}{3}x_1 + \frac{2}{3}x_2 \geq 1$.

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Example (polyhedral core)

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ a_1 + a_2 - 1, & \textit{otherwise.} \end{cases}$$

$$\mathbf{C}(w) = \bigcap_{a \in \{0,1\}^2} C_a(w) = \{x \in [0, 1]^2 \mid x_1 + x_2 = 1\}$$

# Examples of Cores (contd.)

$N = \{1, \ldots, n\}$

Example

$(f_j)_{j \in J}$ ... *family of concave and PH functions* $\mathbb{R}^n \to \mathbb{R}$

$$\boxed{v(a) = \inf \{f_j(a) \mid j \in J\}, \quad \forall a \in \mathbb{R}^n}$$

*The game* $v \restriction [0,1]^n$ *is PH, superadditive, and*

$$\mathbf{C}(v) \neq \emptyset$$

# Core Difficulties

- checking nonemptiness of $\mathbf{C}(v)$ is hard...
- ...even when the core $\mathbf{C}(v)$ is polyhedral:

$$n = 20 \quad \Rightarrow \quad \bigcap_{a \in \{0,1\}^{20}} C_a(v)$$

- a game is played as a one-shot affair: all fuzzy coalitions come up with their demands simultaneously

# Bargaining Schemes

Idea:

- let fuzzy coalitions repeatedly bargain for a final payoff
- capture the bargaining power of individual fuzzy coalitions

**Definition**

*Let $v$ be a game. A bargaining scheme for the core $\mathbf{C}(v)$ is an iterative procedure generating a sequence $(x^k)$ of payoffs converging to $\mathbf{C}(v)$.*

# Enlarged Core

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| $\mathfrak{A}$ | Lebesgue measurable subsets of $[0,1]^n$ |
| $\mu$ | complete probability measure on $\mathfrak{A}$ |

$\mu(A)$ measures the "bargaining power" of the fuzzy coalitions in $A$

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Definition

*Enlarged core of $v$ with respect to $\mu$ is the set*

$$\mathbf{C}_\mu(v) = \bigcup_{\substack{A\in\mathfrak{A}:\\ \mu(A)=1}} \bigcap_{a\in A} C_a(v).$$

Always $\mathbf{C}(v) \subseteq \mathbf{C}_\mu(v)$.

# Characterization of Enlarged Core

$A_x = \{a \in [0,1]^n \mid x \in C_a(v)\}$ coalitions accepting the payoff $x$

**Theorem**

*Let $v$ be a game and $\mu$ be a complete probability measure on $\mathfrak{A}$. If $v$ is Lebesgue measurable, then $A_x \in \mathfrak{A}$ for every $x \in \mathbb{R}^n$, and*

$$\mathbf{C}_\mu(v) = \{x \in \mathbb{R}^n \mid \mu(A_x) = 1\}.$$

**Theorem**

*Let $v$ be a continuous game and $\mu$ be a complete probability measure on $\mathfrak{A}$ such that, for every $A \in \mathfrak{A}$, $\mu(A) > 0$ whenever $A$ is open or $1 \in A$. Then*

$$\mathbf{C}(v) = \mathbf{C}_\mu(v).$$

# Cimmino Type Bargaining Scheme

$P_a x : \mathbb{R}^n \to \mathbb{R}^n$ the projection of $x$ onto $C_a(v)$

The amalgamated projection $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\boxed{\mathbf{P}x = \int_{[0,1]^n} (P_a x)\, d\mu(a), \quad \forall x \in \mathbb{R}^n}$$

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Definition

*The Cimmino type bargaining scheme in the game $v$ is the following rule of generating sequences $(x^k)$ in $\mathbb{R}^n$:*

$$x^0 \in \mathbb{R}^n \quad \textit{and} \quad x^{k+1} = \mathbf{P}x^k, \quad \forall k \in \mathbb{N}_0$$

# Cimmino Type Bargaining Scheme (contd.)

Define

$$\mathbf{g}(x) = \frac{1}{2}\int_{[0,1]^n} \|P_a x - x\|^2 \, d\mu(a), \quad x \in \mathbb{R}^n.$$

**Theorem**

*The mapping* $\mathbf{g}$ *is*

- *nonnegative and everywhere finite*
- *convex*
- *continuously differentiable with* $\nabla \mathbf{g}(x) = x - \mathbf{P}x$

# Cimmino Type Bargaining Scheme (contd.)

**Theorem (Recovering a point in the enlarged core)**

*Let $(x^k)$ be a sequence generated by the Cimmino type bargaining scheme starting from an arbitrary point $x^0 \in \mathbb{R}^n$.*

- *If $(x^k)$ is bounded, then the limit*

$$x^* = \lim_{k \to \infty} x^k$$

*exists, $x^*$ is a minimizer of $\mathbf{g}$ and $\mathbf{g}(x^*) = \lim_{k \to \infty} \mathbf{g}(x^k)$.*

*Moreover, if $\mathbf{g}(x^*) = 0$, then* $\boxed{x^* \in \mathbf{C}_\mu(v)}$

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*Moreover, if $\mathbf{g}(x^*) = 0$, then* $\boxed{x^* \in \mathbf{C}_\mu(v)}$

- *If $(x^k)$ is unbounded or $\mathbf{g}(x^*) \neq 0$, then* $\boxed{\mathbf{C}_\mu(v) = \emptyset}$ *and thus* $\mathbf{C}(v) = \emptyset$.

# Cimmino Type Bargaining Scheme (contd.)

Theorem (Recovering a point in the core)

*Let $v$ be a continuous game and $\mu$ be a complete probability measure on $\mathfrak{A}$ such that, for every $A \in \mathfrak{A}$,*

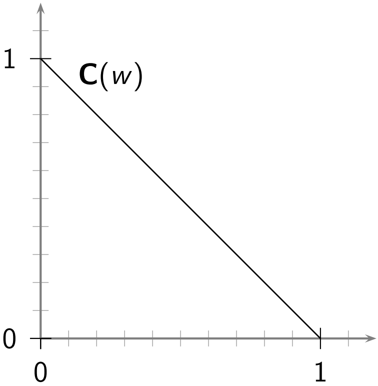
$$\mu(A) > 0 \text{ whenever } A \text{ is open or } 1 \in A.$$

*If $(x^k)$ is a bounded sequence generated by the Cimmino type bargaining scheme with $\mathbf{g}(x^*) = 0$, then*

$$\boxed{x^* \in \mathbf{C}(v)}$$

# Numerical Experiments with Game $w$

$$\mu = \frac{1}{2}\lambda + \frac{1}{2}\delta_1$$



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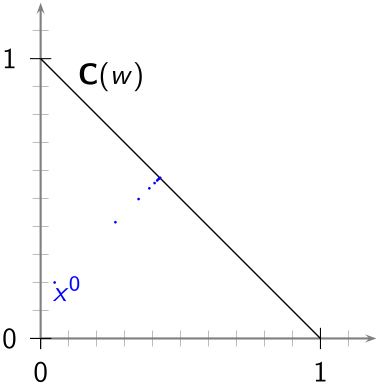
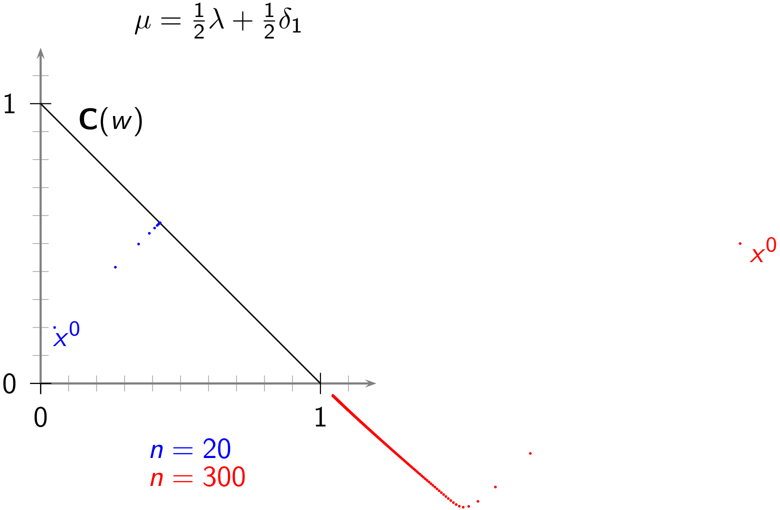


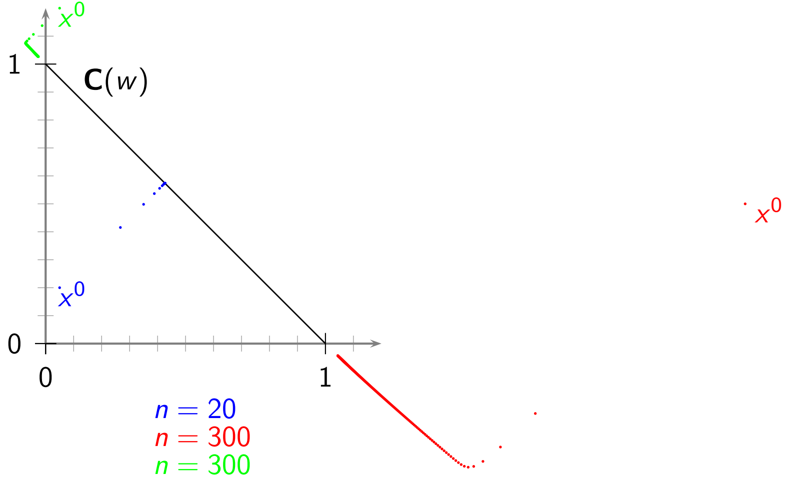

$n = 20$

# Numerical Experiments with Game $w$



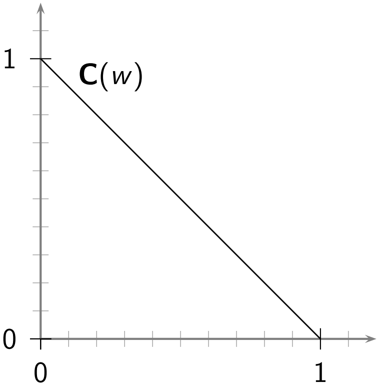
# Numerical Experiments with Game $w$

$\mu = \frac{1}{2}\lambda + \frac{1}{2}\delta_1$ $\qquad g(x^*) \approx 10^{-30}$



# Numerical Experiments with Game $w$ (contd.)

$\mu = \frac{1}{10}\lambda + \frac{9}{10}\delta_1$



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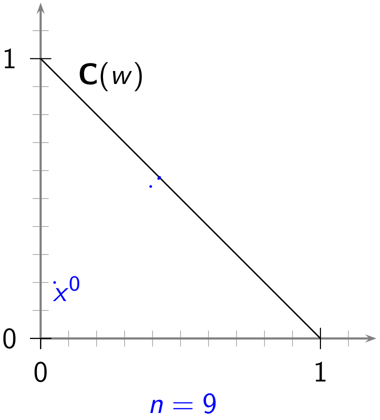

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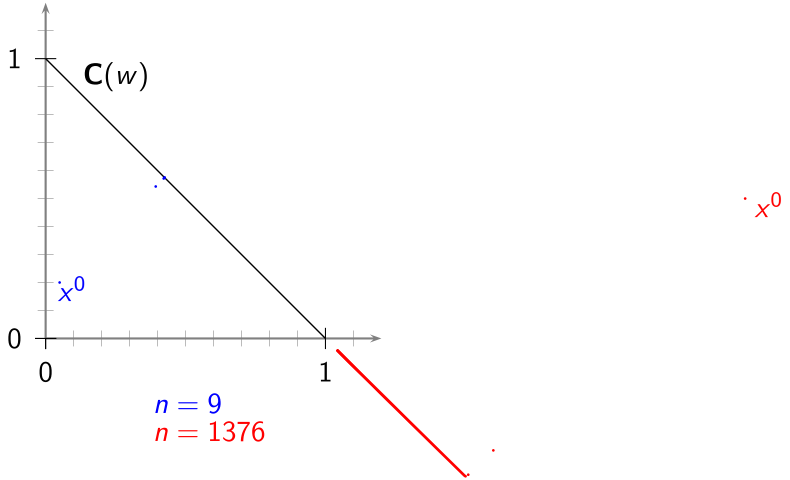

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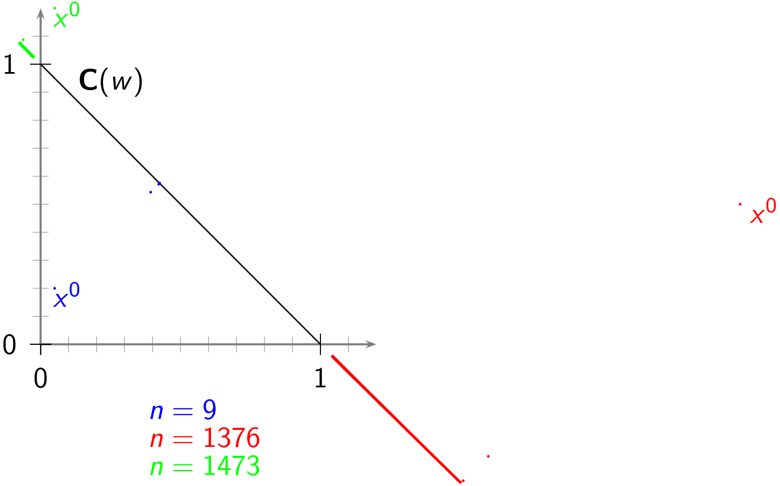
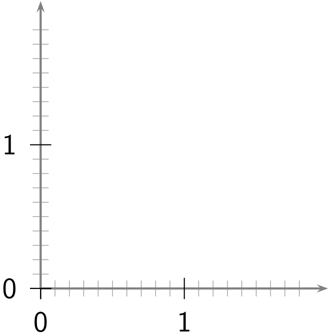


# Numerical Experiments with Game $u$

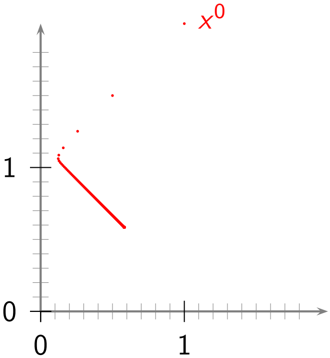
$$\mathbf{C}(u) = \emptyset,\ \mu = \frac{1}{2}\lambda + \frac{1}{2}\delta_1$$



# Numerical Experiments with Game $u$

$\mathbf{C}(u) = \emptyset, \ \mu = \frac{1}{2}\lambda + \frac{1}{2}\delta_1$

$g(x^*) \approx 0.0222879$




$n = 1000$

# Further Directions

- prove that the convergence of Cimmino algorithm is preserved under numerical integration
- accelerate the convergence via relaxed Cimmino algorithm:

$$x^0 \in \mathbb{R}^n \quad \text{and} \quad x^{k+1} = \alpha_k x^k + (1 - \alpha_k)\mathbf{P}_k x^k,$$

  where $(\alpha_k) \in (0, 1]^{\mathbb{N}}$ and $(\mu_k)$ is a sequence of complete probability measures on $\mathfrak{A}$