Finitely Additive Measures

on Algebras of Fuzzy Sets

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Measures on Fuzzy Sets

- roots in theory of fuzzy coalition games (Butnariu, Klement)
- generalization of classical measure theory: σ-**additive** measures on σ-**complete** algebras of fuzzy sets (Mesiar, Navara, ...)
- influence of diverse mathematical theories:
  - many-valued logics
  - quantum and ordered structures
  - operator algebras

Clans

Clan $\mathcal{C}$ over a non-empty set X is a collection of functions $X \longrightarrow [0,1]$ such that

1. $1 \in \mathcal{C}$
2. $\forall f \in \mathcal{C} \;\Rightarrow\; \neg f \in \mathcal{C}$
3. $\forall f, g \in \mathcal{C} \;\Rightarrow\; f \oplus g \in \mathcal{C}$

Łukasiewicz operations

$$f(x) \oplus g(x) := \min(f(x) + g(x), 1)$$
$$f(x) \odot g(x) := \max(f(x) + g(x) - 1, 0)$$

Standard complement $\neg f := 1 - f$

Lattice operations $\vee, \wedge$ (pointwise)

Tribes

Tribe $\mathcal{T}$ is a clan over X such that

$$\forall (f_n) \in \mathcal{T}^{\mathbb{N}} \Rightarrow \bigoplus_{n=1}^{\infty} f_n \in \mathcal{T}.$$

Properties

- **Boolean skeleton** $\mathrm{B}(\mathcal{T}) := \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$ is a σ-algebra
- every function $f \in \mathcal{T}$ is $\mathrm{B}(\mathcal{T})$-measurable

Examples of Clans

- continuous functions $X \to [0, 1]$ on a compact space X
- McNaughton functions
- $\mathcal{A}$-measurable functions $X \to [0, 1]$ on a measurable space $(X, \mathcal{A})$

Separating Clans

Clan $\mathcal{C}$ over X is **separating** if

$$\forall x_1, x_2 \in X \; \exists f \in \mathcal{C} : \; f(x_1) = 0, f(x_2) > 0.$$

Separating clans of continuous functions over a compact subset of $\mathbb{R}$ cannot be tribes!

States and σ-additive States

State s on a clan $\mathcal{C}$ is a mapping $\mathcal{C} \longrightarrow [0, 1]$ such that

1. $s(1) = 1$
2. $\forall f, g \in \mathcal{C}, f \odot g = 0 \;\Rightarrow\; s(f \oplus g) = s(f) + s(g)$

σ-**additive state** on a tribe $\mathcal{T}$ is a state s such that for any non-decreasing sequence $(f_n) \in \mathcal{T}^{\mathbb{N}}$:

$$s\left(\bigvee_{n=1}^{\infty} f_n\right) = \lim_{n\to\infty} s(f_n).$$

Observation The mapping $\mu_s := s \restriction \mathbf{B}(\mathcal{T})$ is a probability measure.

Examples of States

- for a given $x \in X$: $s_x(f) := f(x)$ is a state on any clan
- clan of continuous functions over a compact space X:

$$s(f) := \int_X f d\mu,$$

  where μ is a Borel probability measure

- tribe over X:

$$s'(f) := \int_X f d\nu,$$

  where ν is a probability measure on $\mathbf{B}(\mathcal{T})$

Representation Theorem

Theorem (Butnariu, Klement)
Let $\mathcal{T}$ be a tribe. For any σ-additive state s on $\mathcal{T}$:

$$s(f) = \int_X f d\mu_s.$$

Representation of states on clans

- which clans?
- which representing measures?
- uniqueness?

State Space

$\mathcal{C}$... clan over X

$\mathscr{S}(\mathcal{C})$... collection of all states on $\mathcal{C}$

Theorem The state space $\mathscr{S}(\mathcal{C})$ is a Choquet simplex.

$\mathrm{ext}\ \mathscr{S}(\mathcal{C})$... collection of all extreme points of $\mathscr{S}(\mathcal{C})$

Theorem The following are equivalent:

- $s \in \mathrm{ext}\ \mathscr{S}(\mathcal{C})$
- s is a homomorphism $\mathcal{C} \to [0, 1]$

Description of Extreme Boundary

Theorem

Let $\mathcal{C}$ be a **separating** clan of continuous functions on a compact space X. Then a mapping

$$x \in X \mapsto s_x, \quad s_x(f) = f(x)$$

is a homeomorphism of X onto $\operatorname{ext} \mathscr{S}(\mathcal{C})$.

Convex Analysis

Theorem (Krein,Milman)

If K is a compact convex subset of a LCS X, then $K = \overline{\mathrm{co}}\,\mathrm{ext}\,K$.

Theorem (Integral Representation)

If K is a compact convex subset of a LCS X, then for each $x \in K$ there exists a Borel probability measure ν supported by $\overline{\mathrm{ext}\,K}$ such that for any affine continuous function $\varphi : K \to \mathbb{R}$,

$$\varphi(x) = \int_K \varphi d\nu.$$

In addition, if K is a Choquet simplex and the extreme boundary $\mathrm{ext}\,K$ is compact, then the measure ν is determined uniquely.

Integral Representation 1

Theorem

Let $\mathcal{C}$ be a separating clan of continuous functions over a compact space X and s be a state on $\mathcal{C}$. Then there exists a **unique** Borel probability measure μ such that

$$s(f) = \int_X f \, d\mu, \quad f \in \mathcal{C}.$$

Why?

1. mapping $\hat{f} : s \in \mathscr{S}(\mathcal{C}) \mapsto s(f)$ is an affine continuous function $\mathscr{S}(\mathcal{C}) \to [0,1]$
2. $\hat{f} \circ \varepsilon = f$

Integral Representation 2

If the clan is not separating...

Theorem

Let $\mathcal{C}$ be a clan of continuous functions over a compact space X and s be a state on $\mathcal{C}$. Then there exists **some** Borel probability measure μ such that

$$s(f) = \int_X f d\mu, \quad f \in \mathcal{C}.$$

Counterexample

Let $\mathcal{C}$ be a clan over $[0,1]$ generated by a function

$$f(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ x, & x \in (\frac{1}{2}, 1] \end{cases}$$

The state $s_{\frac{1}{4}}(g) := g(\frac{1}{4})$ is represented by every Dirac measure concentrated between 0 and $\frac{1}{2}$.

References

- Lukeš, J., Netuka, I., Veselý, J. *Choquetova teorie a Dirichletova úloha,* Pokroky matematiky, fyziky a astronomie, 45(2):98–125,2000.
- Phelps, R. R. *Lectures on Choquet's Theorem,* volume 1757 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2nd edition, 2001.

And Next?

- applications to game theory
- curse of $[0, 1]$-valued functions
  $\Rightarrow$ towards formal theory of fuzzy measures