Solving ODE with Fuzzy Initial Condition Using Fuzzy Transform

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Outline

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Fuzzy Partition

Partition of $[a, b]$

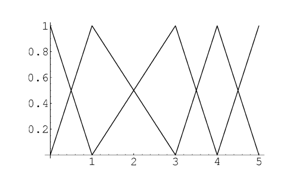
- $a = x_1 < x_2 < \cdots < x_n = b$
- $h(n) = \max_{k=1,\cdots,n-1} (x_{k+1} - x_k)$

Fuzzy Partition of $[a, b]$

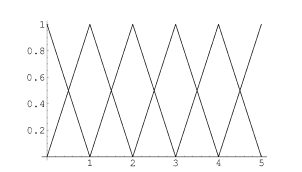
- $A_1(x), ..., A_n(x)$ - basis functions
- $A_k : [a, b] \to [0, 1],\ A_k(x_k) = 1$
- $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where $x_0 = a$ and $x_{n+1} = b$
- A_k is continuous
- $A_k(x)$ increases on $[x_{k-1}, x_k]$ and decreases on $[x_k, x_{k+1}]$
- $\sum_{k=1}^{n} A_k(x) = 1 \quad \forall x \in [a, b]$

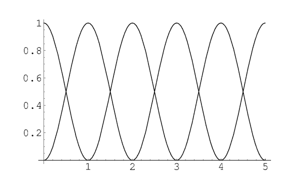
Examples of fuzzy partitions

General fuzzy partition



Uniform fuzzy partitions





F-transform

Definition

Let

- $f \in L^1(a, b)$
- $A_1, \ldots, A_n$ – basis functions on $[a, b]$
- $F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}, \quad k = 1, ..., n$

$[F_1, \ldots, F_n]$ – direct F-transform of f w.r.t. $A_1, \ldots, A_n$

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Inverse F-transform

Definition

- $f \in L^1(a, b)$
- $A_1, \ldots, A_n$ – basis functions on $[a, b]$
- $[F_1, \ldots, F_n]$ – corresponding direct F-transform

The function

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k A_k(x)$$

is called the inverse F-transform.

Convergence of Continuous Functions

Theorem (I. Perfilieva 2001)

Let

- $f \in C([a, b])$,
- $\{A_1^n, A_2^n, \dots A_n^n\}_{n=1}^{\infty}$ – sequence of fuzzy partitions
- $h(n) \to 0$
- $\{f_{F,n}\}_{n=1}^{\infty}$ – corresponding sequence of the inverse F-transforms of f

Then

$$f_{F,n} \rightrightarrows f.$$

Outline

1. F-transform
2. **Generalized Euler method**
3. ODE with fuzzy initial condition

Generalized Euler method

Cauchy problem:

$$y'(x) = f(x, y) \qquad y(x_1) = y_1$$

Direct F-transform

$$\begin{aligned} Y_1 &= y_1 \\ Y_{k+1} &= Y_k + \hat{F}_k \qquad\qquad k = 1, ..., n-1. \end{aligned}$$

$$\hat{F}_k = \frac{\int_a^b f(x, Y_k) A_k(x) dx}{\int_a^b A_k(x) dx}$$

Inverse F-transform

$$y_{Y,n} = \sum_{k=1}^{n} Y_k A_k(x)$$

Outline

1. F-transform
2. Generalized Euler method
3. **ODE with fuzzy initial condition**

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Formulation

Cauchy problem

- $y'(x) = f(x, y)$
- $y(x_1) = \tilde{Y}_1,$ where $\tilde{Y}_1(y_1) = 1$

Two method for modeling of uncertainty development

- multiple solution of ODE with various initial conditions
- using fuzzy relational equations

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Formulation

Cauchy problem

- $y'(x) = f(x, y)$
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Two method for modeling of uncertainty development

- multiple solution of ODE with various initial conditions
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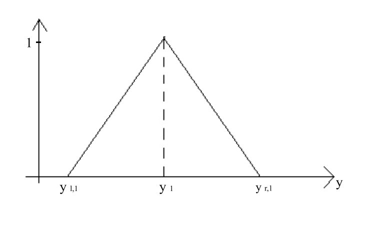
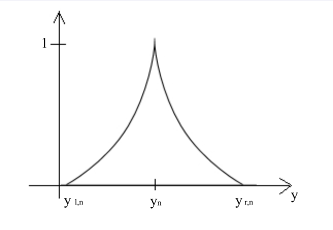
Motivation example



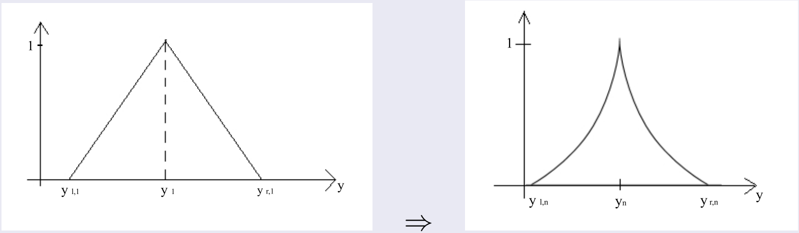
$\Rightarrow$



First method - multiple solution of ODE with various initial conditions

- Advantages - modeling of fuzzy set shape
- Disadvantages - large number of operations

Motivation example



First method - multiple solution of ODE with various initial conditions

- Advantages - modeling of fuzzy set shape
- Disadvantages - large number of operations

First method - multiple solution of ODE with various initial conditions

Algorithm

- partition of interval $[y_{l,1}, y_{r,1}]: y_{l,1} = y_{11} < \ldots < y_{m1} = y_{r,1}$
- mapping of membership degrees: $y_{i1} \mapsto \tilde{Y}_1(y_{i1}) \equiv z_i$
- $\forall y_{i1}, i = 1, \ldots, m$
  - solving Cauchy problem with initial conditions $y(x_1) = y_{i1}$
  $\Longrightarrow$ matrix of values $M \in \mathbb{R}$, $(M)_{ij} = y_{ij}$
- construction of uncertainty development in a nodal point x_j, $j = 1, \ldots, n$:

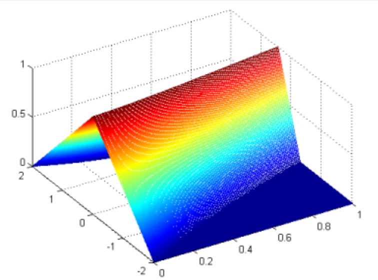
$$\tilde{Y}_j(y_{ij}) = z_i, \quad i = 1, \ldots, m$$

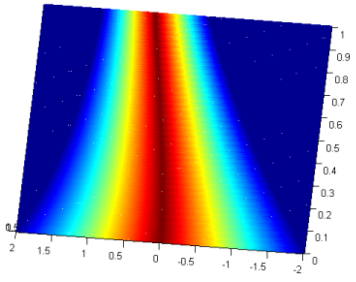
First method

Example

$$y'(x) = \sqrt{x} - y$$

$$y(0) = \tilde{Y}_1, \qquad \tilde{Y}_1(0) = 1$$





Second method - using fuzzy relational equations

Second method - Advantages x Disadvantages

- Advantages - low number of operation
- Disadvantages - low number of information about uncertainty development

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Second method - using fuzzy relational equations

Algorithm

- solving ODE with initial conditions:

$$y(x_1) = y_{l1} \qquad y(x_1) = y_1 \qquad y(x_1) = y_{r1}$$

$$\Longrightarrow y_{l1}, y_{l2}, \dots, y_{ln} \qquad y_1, y_2, \dots, y_n \qquad y_{r1}, y_{r2}, \dots, y_{rn}$$

- creating fuzzy set $\tilde{Y}_k$, $k = 1, \dots, n$ so that

$$\tilde{Y}_k(y_{lk}) = \tilde{Y}_k(y_{rk}) = 0 \qquad \tilde{Y}_k(y_k) = 1$$

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Second method - using fuzzy relational equations

Algorithm

- solving fuzzy relational equation

$$\begin{aligned} A_1 \circ R &= \tilde{Y}_1 \\ \dots\dots\dots\dots &\dots \dots\dots \\ A_n \circ R &= \tilde{Y}_n \end{aligned}$$

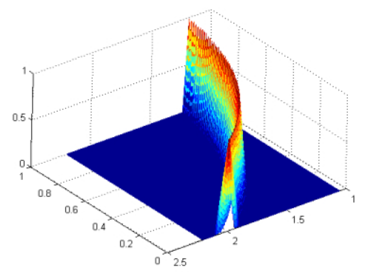
- $\check{R}(x, v) = \bigvee_{i=1}^{n}(A_i(x) * \tilde{Y}_i(v))$
  $R(x, v) = \sum_{i=1}^{n}(A_i(x) \cdot \tilde{Y}_i(v))$
- uncertainty development is given by $R(\underline{x}, v)$ in a point $\underline{x} \in [a, b]$

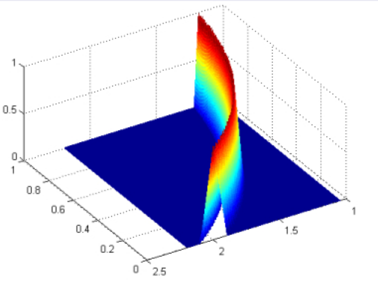
Example

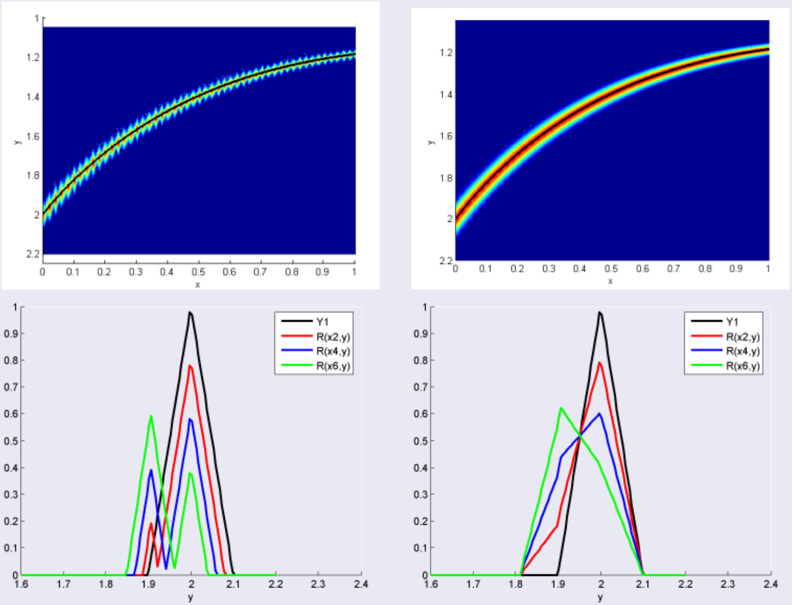
Example

$$y'(x) = \sqrt{x} - y$$

$$y(0) = \tilde{Y}_1, \qquad \tilde{Y}_1(2) = 1$$






y
x
y
x
Y1
R(x2,y)
R(x4,y)
R(x6,y)
y
Y1
R(x2,y)
R(x4,y)
R(x6,y)
y

Thank you for your attention