# Compenzational Vagueness

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**Abstract**

Some manipulations with vague quantities consist in an aggregation of vague amounts where the resulting aggregated quantity (in our case a sum of vague summands) is expected to be (almost) crisp. The aim of this contribution is to analyze and briefly discuss various approaches to this problem which can be supported by the elementary theory of fuzzy quantities. The analysis is focused on the possibility of achieving the desired sum, on the fuzzy set theoretical methods applicable to this model, and also to the limits of regulation of some fuzzy summands during the aggregation process.

**Keywords:** Fuzzy quantity, extension principle, decomposition paradigm, shape, scale, compensation of vagueness, vagueness.

## 1 Introduction

The well handled processing of fuzzy quantities appears to be one of the most prospective trends of further development of fuzzy set theory and applications. It becomes important for advanced methods of management of uncertainty and vagueness in numerous economic, sociological and organizational procedures. The significancy of the multilevel investigation of fuzzy quantities is analyzed from the point of view of fuzzy methodology (see, e. g., [12, 13, 14]), its traditional methods are modified to be more adequate to the real problems (e. g., in [11]), and the fuzzy set theoretical model of quantitative phenomena is widely generalized (as, e. g., in [1]).

This development of the general theory implies an extension of the scale of its potential applications to new sorts of problems. In this paper, we discuss several approaches to the problem of aggregation of vague (it means fuzzy) components aiming to minimize the variability (fuzziness) of the result.

Such problem seems to be rather contradictoric with regard to the essence of fuzziness, nevertheless, it is not as unusual as it appears.

Let us consider, for example, a model of transport line passing one or more check-points where the time demanded for passing particular intervals is relatively free, meanwhile their total is given by a time-table as a narrow interval. Very similar situation appears when a container is part-wise filled by some material, where particular doses are in certain limits arbitrary but the total amount is given quite strictly. All such situations, formulated in the language of fuzzy set theory, mean the demand to get a crisp (or almost crisp) sum of fuzzy components. This is impossible in the classical model based on the extension principle. However, such situations exist and they are not very rate.

Some attempts to it were done in [7, 8], some others can be potentially derived by means of the decomposition model (see [11] and others). In this contribution, we discuss the effectivity and adequacy of some of them. Namely, we compare the models using extension and decomposition principles. The questions on which we focuse our attention are:

— how to estimate the possibility that (after the realization of particular vague additions) the total sum will be equal or very near to the desired value,

— how to characterize the possibilities of compensation of declinations of consequently added elements,

— which of both principles of computation with fuzzy quantities appears more effective (and due to which criteria) in modelling the real situations of the considered type.

## 2 Fuzzy Quantity

In the whole paper, we denote by $R$ the set of all real numbers.

Due to [5, 6] and other works, we define *fuzzy quantity* $\boldsymbol{a}$ as a fuzzy subset of $R$ with membership function $\mu_a$ such that:

(1) There exists $x_a \in R$ such that $\mu_a(x_a) = 1$.

(2) There exist $x_1, x_2 \in R$ such that $x_1 < x_a < x_2$ and $\mu_a(x) = 0$ for $x \notin [x_1, x_2]$.

Any member $x_a \in R$ fulfilling (1) is called a *modal value* of $\boldsymbol{a}$. The set of all fuzzy quantities will be denoted by $\mathbb{R}$. The properties of fuzzy quantities are summarized in many works. Let us mention at least [2, 3, 4, 5, 6, 9].

To simplify the notations, we denote by $\langle r \rangle$, where $r \in R$, the fuzzy quantity in $\mathbb{R}$ which is condensed in a single possible value, i. e.

$$(3) \qquad \mu_{\langle r \rangle}(r) = 1, \quad \mu_{\langle r \rangle}(x) = 0 \quad \text{for } x \neq r.$$

If $\boldsymbol{a} \in \mathbb{R}$ then we denote by $(-\boldsymbol{a}) \in \mathbb{R}$ the *opposite* fuzzy quantity to $\boldsymbol{a}$, and

$$(4) \qquad \mu_{(-a)}(x) = \mu_a(-x) \quad \text{for all } x \in R.$$

As the fuzzy quantities are, in fact, extensions of crisp real numbers, it is rational to extend even the algebraic operations used over $R$, on $\mathbb{R}$. It is usually done by so called *extension principle*. In this paper we need only the operation of addition which, extended over $\mathbb{R}$, is denoted by $\oplus$, and if $\boldsymbol{c} = \boldsymbol{a} \oplus \boldsymbol{b}$, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}$, then

$$(5) \qquad \mu_c(x) = \sup_{y \in R} [\min(\mu_a(y), \mu_b(x - y))]$$

for any $x \in R$. The properties of this (and others) operation are summarized, e. g., in [2, 3, 5, 6]. It can be easily seen that they are rather similar but not identical with the group properties of the addition of crisp real numbers.

Some of the essential differences between both addition operations are related with the concept of *fuzzy zero*. Its seemingly natural definition as $\langle 0 \rangle$ (note, $\boldsymbol{a} \oplus \langle 0 \rangle = \boldsymbol{a}$ for every $\boldsymbol{a} \in \mathbb{R}$) does not fit with the concept of opposite element as $(-\boldsymbol{a})$. As shown in [5, 6] and further specified in [9], fuzzy zeros form a class of fuzzy quantities, closed for the operation $\oplus$, which we denote

$$(6) \qquad \mathbb{S} = \{\boldsymbol{s} \in \mathbb{R} : \mu_s(x) = \mu_s(-x) \text{ for all } x \in R\}.$$

Fuzzy quantities from $\mathbb{S}$ are called *symmetric*. Due to [9], the class $\mathbb{S}$ can be narrowed to

$$(7) \qquad \mathbb{Z} = \{\boldsymbol{z} \in \mathbb{R} : \boldsymbol{z} \in \mathbb{S}, \, \mu_z(0) = 1\},$$

whose elements are called *strongly symmetric*. The algebraic properties of symmetric and strongly symmetric fuzzy quantities are equivalent and they agree with the group axioms (including commutativity) under the condition that we substitute the identity between fuzzy quantities $a = b$, $a, b \in \mathbb{R}$,

$$(8) \qquad \boldsymbol{a} = \boldsymbol{b} \quad \text{if} \quad \mu_a(x) = \mu_b(x) \quad \text{for all } x \in R,$$

by a weaker equivalence due to which $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}$ are equivalent iff they differ in fuzzy zeros, only (see, [5, 6, 9]).

The comparison and ordering relation between fuzzy quantities can be (and often is) defined in different ways, due to the character of the solved problem (a representative analysis of it can be found in [4], some possibilities are mentioned in [6], too). Here, we use the ordering relation most respecting the vague structure of fuzzy quantities. Namely, we define the *fuzzy ordering* relation $\succeq$ over $\mathbb{R}$ as a fuzzy relation with membership function $\nu_{\succeq} : \mathbb{R} \times \mathbb{R} \to [0, 1]$ such that for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}$ the possibility of $\boldsymbol{a} \succeq \boldsymbol{b}$ is

$$(9) \qquad \nu_{\succeq}(\boldsymbol{a}, \boldsymbol{b}) = \sup_{\substack{x, y \in R \\ x \geq y}} [\mu_a(x), \, \mu_b(y)],$$

and, analogously, we define *fuzzy equality* $\boldsymbol{a} \sim \boldsymbol{b}$ as fuzzy relation with $\nu_{\sim}(\boldsymbol{a}, \boldsymbol{b}) : \mathbb{R} \times \mathbb{R} \to [0, 1]$,

$$(10) \qquad \nu_{\sim}(\boldsymbol{a}, \boldsymbol{b}) = \sup_{x \in R} [\mu_a(x), \, \mu_b(x)].$$

## 3 Decomposition Paradigm

The algebraic operations based on the extension principle, including the addition (5), are natural. They reflect the expected properties of composition of vagueness of summands after the summation procedure in the standard situations. One of the evident results is, that the vagueness of the sum is higher than the one of the summands, where the vagueness is represented by the extent and values of the membership functions. This observation contradicts with the essential demand of the problem analyzed in this paper. Hence, it appears evident that the extension principle cannot serve as the method of its solution.

An alternative approach to the processing of fuzzy quantities increasing the flexibility of actually applied methods was suggested in several papers, including [7], and it was summarized in [11]. It is based on so called *decomposition paradigm* due to which any fuzzy quantity $\boldsymbol{a} \in \mathbb{R}$ consist of three components:

— the *crisp numerical value* $x_a$

— the *shape* function $\varphi_a$ characterizing the structure of vagueness connected with $\boldsymbol{a}$,

— the increasing *scale* function $f_a$ characterizing the structure of vagueness connected with the source of data.

The formal properties of all these components are presented, e. g., in [11]. The essential meaning of this decomposed model is that it admits the possibility to process each of the components separately, with regard to the demands of actual application. Such processing leads to the component of the resulting fuzzy quantity (let us denote it $\boldsymbol{c} \in \mathbb{R}$ with components $(x_c, \varphi_c, f_c)$), where the membership function $\mu_c$ is derived by means of

$$\mu_c(x) = \varphi_c\left(f_c(x) - f_c(x_c)\right), \quad x \in R. \tag{11}$$

The consequence of the decomposition paradigm significant for the model of addition which we are looking for is the admissibility of separate processing of shapes independent on the scales and crisp values. The shapes represent the main features of the membership functions – namely, they reflect the main structure of vagueness and its behaviour. In the following sections, we use especially the possibility to combine the shapes not only by means of the extension principle (or its very close analogy) but by other relevant fuzzy set theoretical operations. The choice of the one used in a specific model can be modified with respect to the described reality.

# 4 Composition of Vagueness

In the main section, we briefly discuss and compare some approaches to the problem formulated in the Introduction.

Let us consider fuzzy quantities $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c} \in \mathbb{R}$ with membership functions $\mu_a$, $\mu_b$, $\mu_c$, respectively. Let us denote, for any $\boldsymbol{d} \in \mathbb{R}$ with $\mu_d$, the values

$$\mu_d^{\sup} = \sup\{x \in R : \mu_d(x) > 0\} \tag{12}$$
$$\mu_d^{\inf} = \inf\{x \in R : \mu_d(x) > 0\} \tag{13}$$

and

$$V_d = x_d^{\sup} - x_d^{\inf}. \tag{14}$$

In the following subsections, we analyze the demand that

$$\min(V_a, V_b) > V_c \tag{15}$$

including the possibility that $\boldsymbol{c} = \langle x_c \rangle$ for modal value $x_c$ of $c$. By $x_a$, $x_b$ we denote some of the (possibly many) modal values of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

The main goal of this contribution is to discuss some of the possible compositions of $\boldsymbol{a}$ and $\boldsymbol{b}$ whose result is near to the fuzzy quantity $\boldsymbol{c}$. For the purpose of general problem discussion, we denote this composition by $\circ$, which means that we discuss the properties of the relations represented by formulas

$$\boldsymbol{a} \circ \boldsymbol{b} \sim \boldsymbol{c} \quad \text{or} \quad \boldsymbol{a} \circ \boldsymbol{b} \succeq \boldsymbol{c}. \tag{16}$$

Special attention is focused on the compositions of fuzzy quantities $\boldsymbol{a} \circ \boldsymbol{b}$ where (15) is fulfilled.

In the following subsections, we briefly discuss three specific cases of the composition operation $\circ$, especially from the point of view of the degree, in which they respect demands (15) and (16). Priority is given to the following questions:

— How large is the possibility that the composition $\boldsymbol{a} \circ \boldsymbol{b}$ is close to $\boldsymbol{c}$?

— What is the extent of possible values of the composed fuzzy quantity $\boldsymbol{a} \circ \boldsymbol{b}$ (i. e., the value of $V_{a \odot b}$)?

— Does the actual method include a possibility of compensation of the composition by means of proper modification of fuzzy quantity $\boldsymbol{b}$ in accordance with $\boldsymbol{a}$?

— Is it possible to compare the effectivity of particular composition methods.

## 4.1 Extension Principle Model

The first model of composition operation which we consider, is the classical extension principle (5), i. e., we consider the operation $\oplus$ in the position of $\circ$. It means that we compute, by means of the extension principle, the sum $\boldsymbol{a} \oplus \boldsymbol{b} \in \mathbb{R}$ with membership function $\mu_{a \oplus b}$.

Due to the above notation

$$\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a} \oplus \boldsymbol{b}. \tag{17}$$

Unfortunately, this method evidently does not respect the most important general demand, namely the one demanding (see (14))

$$V_{a \oplus b} < \min(V_a, V_b).$$

**Observation 1.** The possibility of $\boldsymbol{a} \oplus \boldsymbol{b} \sim \boldsymbol{c}$ is equal to 1 iff there exist modal values $x_a$, $x_b$, $x_c$ of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, respectively, such that

$$x_a + x_b = x_c.$$

**Observation 2.** The possibility that $\boldsymbol{a} \oplus \boldsymbol{b} \sim \boldsymbol{c}$ is generally equal to $\mu_{\sim}(\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{c})$ due to (10).

**Observation 3.** Using (12), (13) and (14),

$$x_{a\oplus b}^{\inf} = x_a^{\inf} + x_b^{\inf}, \quad x_{a\oplus b}^{\sup} = x_a^{\sup} + x_b^{\sup},$$

hence

$$V_{a\oplus b} \geq \max(V_a, V_b) > V_c.$$

**Observation 4.** The previous statements cannot be changed by any choice of $\boldsymbol{b}$. If we choose $\boldsymbol{b} = \langle x_c - x_a \rangle$ as a crisp quantity then $V_{a\oplus b} = V_a > V_c$, and $\nu_{\sim}(\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{c}) = 1$.

The previous observations show that the dogmatic application of the extension principle does not generally aim to the desirable model of the considered situation.

## 4.2 Complementation Model

The next model modifies the previous extension principle approach by accenting stronger stress on the potential possibility of compensation of vague declinations of the first component by proper modification of the second one.

Evidently, the space for the choice of the second summand (we are used to denote it $\boldsymbol{b} \in \mathbb{R}$) is not unlimited. Its boundaries are often given by physical or technological parameters, accessible stock of material or energy, qualification and accessibility of human resources, etc. Let us denote the *accessible set of the second summands* by $\mathbb{B} \subset \mathbb{R}$.

Using the above notation, the operation $\circ$ in this case means

$$\text{(18)} \qquad \boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a} \oplus \boldsymbol{d} \quad \text{for some } \boldsymbol{d} \in \mathbb{B},$$

or, more generally,

$$\boldsymbol{a} \circ \boldsymbol{b} = \{\boldsymbol{a} \oplus \boldsymbol{d} : \boldsymbol{d} \in \mathbb{B}\}.$$

The choice of $\boldsymbol{d} \in \mathbb{B}$ is to respect the following statements.

Summarizing the model, we can see that there exists a desirable terminal fuzzy quantity $\boldsymbol{c} \in \mathbb{R}$ which is to be as exact as possible, with a modal value $x_c$. Moreover, "the first" summand, a fuzzy quantity $\boldsymbol{a} \in \mathbb{R}$ is (vaguely) known. Our aim in this subsection is to choose "the second" summand, fuzzy quantity $\boldsymbol{b} \in \mathbb{B} \subset \mathbb{R}$ which is the best one from the point of view of the goal of this model. Let us suppose, still, that the considered composition operation $\circ$ is the extension principle summation $\oplus$. Then all observations introduced in Subsection 4.1 keep valid, and, moreover, the following ones can be seen whenever (18) is kept in mind.

**Observation 5.** If there exists $\boldsymbol{b} \in \mathbb{B}$ such that $x_b = x_c - x_a$ then $\nu_{\sim}(\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{c}) = 1$ and, consequently, $\nu_{\sim}(\boldsymbol{a} \circ \boldsymbol{b}, \boldsymbol{c}) = 1$.

**Observation 6.** If the fuzzy quantity condensed in one possible value $\langle x_c - x_a \rangle \in \mathbb{B}$ and if we put $\boldsymbol{b} = \langle x_c - x_a \rangle$ then the statement of Observation 5 holds and, moreover $V_{a+b} = V_a$ which extent of variability is the nearest one to $V_c$.

**Observation 7.** If we put $\boldsymbol{b} = \boldsymbol{c} \oplus (-\boldsymbol{a})$ and if $\boldsymbol{b} \in \mathbb{B}$ then it fulfils assumptions of Observation 5.

## 4.3 Accessibility Testing Model

One of the questions to be answered whenever we try to solve the problem of (almost) deterministic total value of vague summands is the possibility that at least one value of the sum $\boldsymbol{a} \circ \boldsymbol{b}$ is equal to the desired total value $\boldsymbol{c}$.

Let $\boldsymbol{a}$, $\boldsymbol{b}$ be given, and let us define $\boldsymbol{d} = \boldsymbol{c} \oplus (-\boldsymbol{a})$. Then the eventual possibility of successful outcome of our problem, it means the possibility that

$$\boldsymbol{a} \oplus \boldsymbol{b} \sim \boldsymbol{c},$$

is a fuzzy phenomenon whose possibility is

$$\text{(19)} \qquad \nu_{\sim}(\boldsymbol{d}, \boldsymbol{b}).$$

In a more lucid way, this possibility can be defined also by the formula

$$\text{(20)} \qquad \max_{x\in R} [\min(\mu_d(x), \mu_b(x))].$$

**Observation 8.** Analogously to Observation 1, possibilities (19) and (20) are equal to 1 iff there exist modal values $x_a$, $x_b$, $x_c$ such that $x_a + x_b = x_c$, as in such case also $x_d = x_c - x_a = x_b$ and, consequently $x_c = x_a + x_d$, as well.

# 5 Extent of Vagueness

The problem of regulation and compensation of summands in order to maximize the possibility that the sum will be close to some not very extensive fuzzy quantity has also another aspect. In the previous section, the main attention is focused on the conditions, under which at least the modal values of the sum and goal quantity are identical. The second problem, to achieve the vagueness of the sum as narrow as possible, demands a qualitatively new approach based on the methods suggested (and analyzed) in [11, 7] and in several related papers. It means, to process the crisp modal values and the shapes of the vagueness separately, and by means of other procedures than the extension principle.

Let us consider fuzzy quantities $\boldsymbol{a}$, $\boldsymbol{b}$ with modal crisp values $x_a$, $x_b$ and shape functions $\varphi_a$, $\varphi_b$, respectively (cf. Section 2). To simplify the notations, we put the scales $f_a$, $f_b$ (see Section 2 and (11))

$$f_a(x) = f_b(x) = x, \tag{21}$$

and, consequently, it is possible to construct the membership functions $\mu_a$, $\mu_b$, using (11)

$$\mu_a(x) = \varphi_a(x - x_a), \quad \mu_b(x) = \varphi_b(x - x_b), \tag{22}$$

where the shape functions fulfil:

— $\varphi_a(x) \in [0,1]$, $\varphi_b(x) \in [0,1]$, $x \in R$,

— $\varphi_a(0) = \varphi_b(0) = 1$,

— $\varphi_a$, $\varphi_b$ are continuous, not decreasing for $x < 0$, and not increasing for $x > 0$,

— there exist $x_1$, $x_2 \in R$ such that $\varphi_a(x) = \varphi_b(x) = 0$ for $x \notin [x_1, x_2]$.

Evidently, the shape functions represent normalized forms of the membership functions.

Respecting the procedures, methods and conclusions discussed in [11], we may compute a composition $\boldsymbol{a} \circ \boldsymbol{b}$, due to Subsection 4.1, by means of separate composition of $x_a$, $x_b$ and $\varphi_a$, $\varphi_b$. Let us keep the denotation

$$\boldsymbol{c} = \boldsymbol{a} \circ \boldsymbol{b}, \quad \boldsymbol{c} \text{ is defined by } (x_c, \varphi_c) \tag{23}$$

where $\varphi_c$ fulfils the above properties of the shapes and the eventual scale $f_c$ of $\boldsymbol{c}$ is equal to $f_a$ and $f_b$.

In the case of our problem, it is natural to combine (or compose) fuzzy quantities in (23) by

$$x_c = x_a + x_b \quad \text{and} \quad \varphi_c = \varphi_a \circ \varphi_b, \tag{24}$$

where the composition of shapes $\varphi_a \circ \varphi_b$ can be relatively arbitrary, under the condition that it reflects the natural character of the modelled reality.

In our case, two following procedures appear natural and adequate to the problem:

$$\varphi_c(x) = \min\left(\varphi_a(x), \varphi_b(x)\right), \quad x \in R, \tag{25}$$

and

$$\varphi_c'(x) = \varphi_a(x) \cdot \varphi_b(x), \quad x \in R, \tag{26}$$

however, the possibility of other, more sophisticated, operations is admissible as well.

**Observation 9.** Evidently, $\mu_c(x) = \varphi_c(x - x_c)$ fulfills $\mu_c(x_c) = 1$ and

$$V_c = V_a \cap V_b,$$
$$\mu_c(x) = \min\left(\mu_a(x - x_b), \mu_b(x - x_a)\right), \quad x \in R,$$

which means that the demand of the minimization of the extent of the vagueness is fulfilled.

**Observation 10.** Similarly, if (26) is used and if $x_1 > -1$, $x_2 < 1$ then

$$V_c \subset V_a \cap V_b,$$
$$\mu_c(x) \le \min\left(\mu_a(x - x_b), \mu_b(x - x_a)\right), \quad x \in R,$$

which means that the goals of the procedure are fulfilled, as well.

**Observation 11.** If $x_1 < -1$, or $x_2 > 1$, then the extents of vagueness $V_c$, $V_a$, $V_b$ do not fulfill the previous Observation 10 but, the significantly large values of $\mu_c(x)$ are condensed near $x_c$.

# 6 Conclusive Remarks

In the last two sections, we have analyzed the problem of construction of an additive composition method resulting into a fuzzy quantity with minimal uncertainty, from two points of view. How to maximize the possibility of the most desired value, and how to minimize the uncertainty connected with the result. The second condition is very near to the demand to decrease the possibility of less desired values of the result.

In fact, the optimal approach to the problem formulated in Introduction is based on s simultaneous application of both views and by combination of their results. This represents the possible continuation of the research introduced here.

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