Arithmetic Circuits of the Noisy-Or Models

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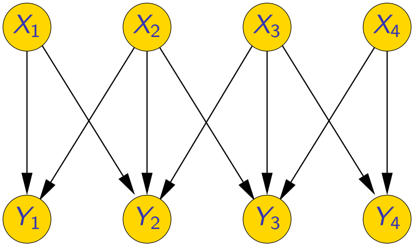

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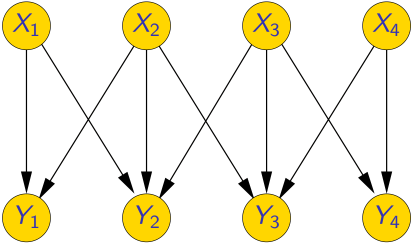

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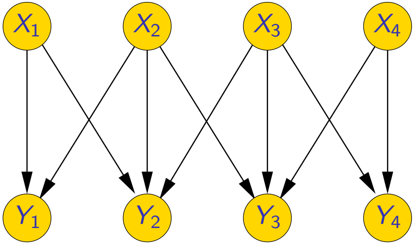

Example

The nodes from the first level represent system faults and the nodes from the second level the observations.

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- p_j is called the inhibition probability of $X_j = 1$ since $Y = 0$ only if all its parents with value 1 are inhibited.

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Example - deterministic or

CPT $P(Y|X_1, \ldots, X_n)$ represents an or gate if $p_j = 0$ for all $j \in \{1, \ldots, n\}$.

$$
\begin{aligned}
P(Y = 0 | X_1 = x_1, \ldots, X_n = x_n) &= \prod_{j=1}^{n} 0^{x_j} \\
&= \begin{cases} 1 & \text{if } x_j = 0 \text{ for } j \in \{1, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}
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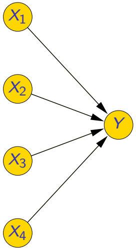
Compilation of a noisy-or gate - the standard BN approach

Lauritzen and Spiegelhalter (1988), Jensen et al. (1990), Shafer and Shenoy (1990)



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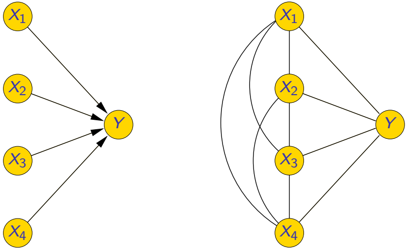

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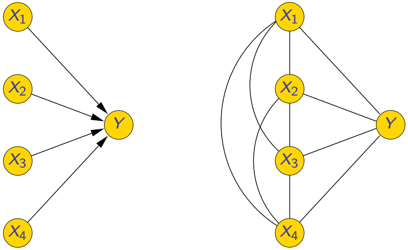

The total table size is $2^5 = 32$.

Compilation of a noisy-or gate - parent divorcing

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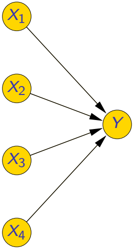

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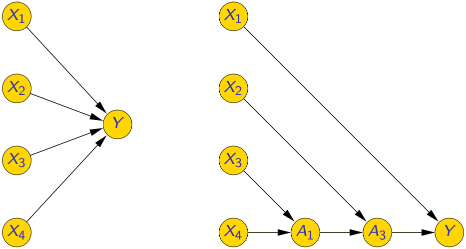

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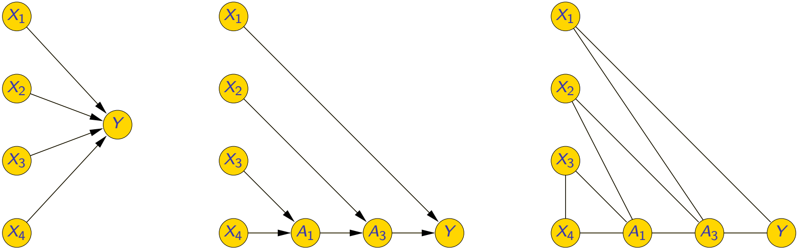

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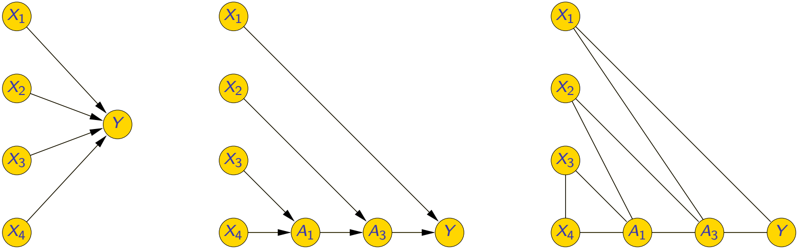

The total table size is $3 \cdot 2^3 = 24$.

Rank-one decomposition

Díez and Galán (2003), Vomlel (2002), Savický and Vomlel (2007)

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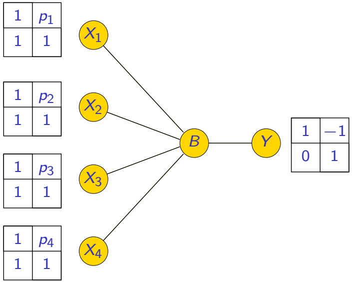
Rank-one decomposition (example)



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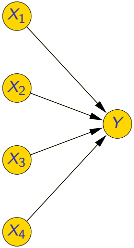

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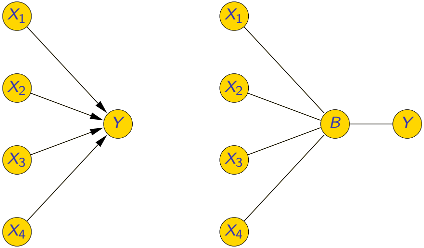

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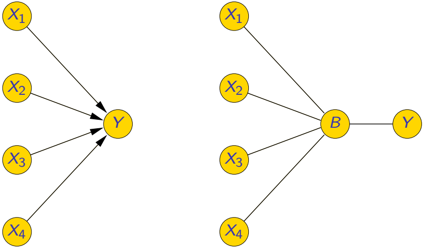
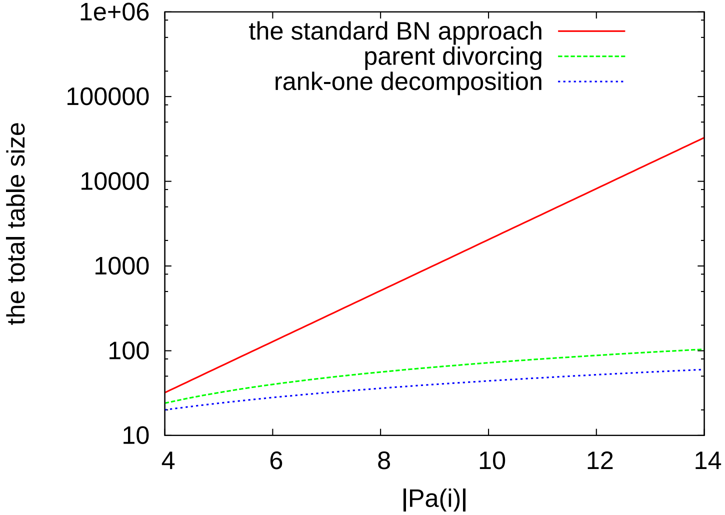


The total table size is $5 \cdot 2^2 = 20$.

Comparisons for the noisy-or gate



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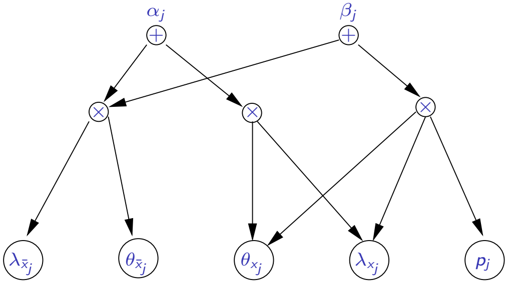
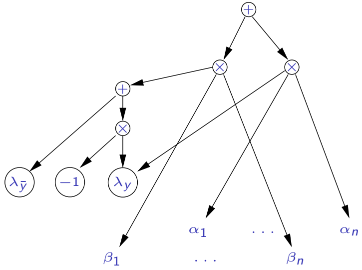
Circuit output:

- probability of evidence $P(\mathbf{e})$.

AC of a noisy-or gate





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- An AC may also represent more efficient computations due to specific properties of the initial BN (e.g., determinism, context specific independence).
- The size of an AC (i.e. number of its edges) can be used as a measure of inference complexity

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- Ace uses the parent divorcing method for preprocessing noisy-or models.
- We use the size of ACs to compare the effect of preprocessing Bayesian networks by Ace's parent divorcing giving (what we call) the original model and by rank-one decomposition giving the transformed model.

Experiments

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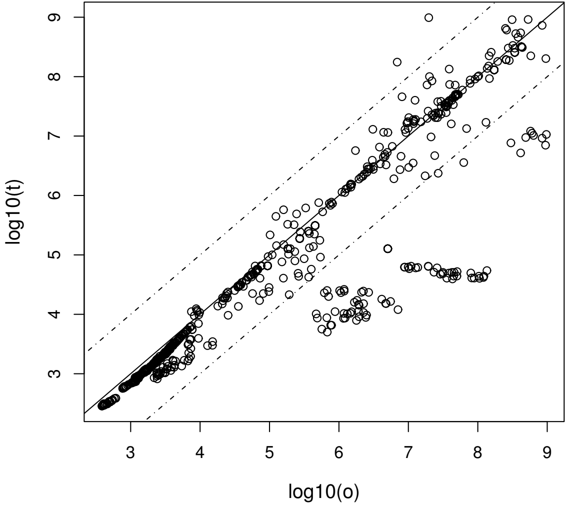
- For every node from the bottom level we randomly selected `e/y` nodes from the top level as its parents.

Transformed vs. original model AC size



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- In several cases we got significant reductions in the AC size - in a few cases multiple order of magnitude.
- There are also eleven cases where the AC of the transformed model is at least three times larger - we will comment on these cases on the next slide.

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- However, if a heuristic triangulation method is used then it may happen that we get larger treewidth for the triangulated graph of the transformed model.
- We conducted additional experiments with all eleven models with significant loss in the AC size.
- In all of these eleven cases we were able to get the AC of the transformed model smaller than the AC of original model using the triangulation derived from the triangulation of the original model.