FUZZY COMPONENTS OF COOPERATIVE MARKET*

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Abstract. The models of the free exchange market belong to the basic contributions of mathematics and, especially, operations research to the theoretical investigation of economic phenomena. In this chapter we deal with the Walras equilibrium model and its more cooperative modification and analyze some possibilities of its fuzzification. The main attention is focused on the vagueness of utility functions and of prices, which can be considered for most subjective (utilities) or most unpredictable (prices) components of the model. Some marginal comments deal with the sense and possibility of fuzzification of the cooperating coalitions. The elementary properties of the fuzzified model are presented and the adequacy of the suggested fuzzy set theoretical methods to the specific properties of real market models is briefly discussed.

1 Introduction

The models of market equilibrium, here we deal especially with Walras equilibrium, are deeply investigated in the literature on operations research. Their close relation to the cooperative game theory belongs to most significant results of that investigation, and its different modifications are still thoroughly analyzed.

The classical market models are deterministic, respecting the paradigm that all input parameters are exactly know. It is evident that this presumption is not correct in many real situations in which the exchange of goods (in a very wide sense) is realized. Namely a market as an environment in which subjective preferences, intuitive expectations and rather chaotic behavior of individual agents are typical phenomena, is an object of investigation in which some vagueness is to be expected and included in the theoretical models. It means that the substitution of some of its components by their fuzzy counterparts is quite desirable. Most of this chapter is devoted to the fuzzification of two quantitative data – namely, the individual utilities and the prices – and to their representation by fuzzy quantities (see, e. g., [10, 11]). In this respect, the fuzzified cooperative game model presented in [12] and developed in some other papers, appears to be a useful analogy.

Moreover, some sections of this work freely use some ideas which were briefly developed in [7] and [8], and which regard the more intensive cooperation among participants of the market, where there exist some groups of agents behaving like homogeneous blocks

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respecting the standard market behaviour if we consider their market activities towards other partners, but whose members among themselves use much more liberal limitations of their exchange (e. g., they do not respect prices) in order to maximize the total profit of the block.

The cooperative behaviour is, on a general level, modelled by so called cooperative games. For our purposes, we focus our attention on the games with transferable utility, briefly TU-games, (see, e. g. [6, 16, 17]). Their close relation to the market models is well known and it is formulated, e. g., in [5]. It means that even the fuzzification of the market and its equilibrium can be inspired by well elaborated approaches to the fuzzification of TU-games. One of them consisting in the fuzzification of some quantitative components of the model, was already mentioned in one of the above paragraphs, and it is investigated in Section 3 of this chapter. Its main concepts are analogous to fuzzy cooperative games model [12], use the methodology summarized in [10] and [11], and further develop the model briefly suggested in [9]. There exists also another approach to the fuzzification of TU-games, consisting in the fuzzification of the structure of the cooperation. For the games, it was formulated in [1, 2], and it is further developed (cf. [3, 4, 13, 14]) till now. Its transformation to the market model is not simple. It demands deep analysis of the particular forms of participation of agents (players) in coalitions, especially in the cases where their market activities are to be distributed among several coalitions. In Section 4 of this paper, we briefly discuss this topic from the point of view of its eventual development in other papers.

The fuzzification of the market equilibrium suggested here opens new field of investigation which appears to be perspective and which can lead to some inspirative results on the behaviour of agents in free exchange markets being connected with uncertainty and vagueness typical for realistic market situations.

2 Preliminaries

Before the presentation of the analyzed model, it is useful to recollect some concepts which are used in the following sections. They regard, especially, the theory of fuzzy quantities, deterministic cooperative games with transferable utility and deterministic market equilibria.

In this and all following sections we denote by R the set of all real numbers, by R^m we denote the set of m-dimensional real vectors and by R_+, R_+^m their subsets of non-negative components. Moreover, if M is a set then we denote by $\mathcal{F}(M)$ the set of all fuzzy subsets of M.

2.1 Fuzzy Quantities

The vague quantitative components of the models presented below are characterized by fuzzy quantities. Due to [10, 11] and other works, *fuzzy quantity* is any fuzzy subset a of R, i. e. $a \in \mathcal{F}(R)$, with membership function $\mu_a(R) \to [0, 1]$ such that

— there exists $r_0 \in R$ such that $\mu_a(r_0) = 1$, (r_0 is called *modal value* of a)

— the support set of μ_a is limited.

Fuzzy quantities represent vague numerical values and they can be processed analogously to their deterministic counterparts. For our purposes, we need the algebraical properties of summation and multiplication by crisp real number and the relation of ordering over the set of fuzzy quantities.

If a, $b \in \mathcal{F}(R)$ are fuzzy quantities with membership functions μ_a, μ_b, respectively, then fuzzy quantity $a \oplus b$ with $\mu_{a\oplus b} : R \to [0,1]$, where for $r \in R$

$$\mu_{a\oplus b}(r) = \sup_{s\in R} \left[\min(\mu_a(s),\, \mu_b(r-s))\right], \tag{1}$$

is called *sum* of a and b. Moreover, if $r \in R$, then the fuzzy quantity $r \cdot a$ with $\mu_{r\cdot a} : R \to [0,1]$ such that for $s \in R$

$$\begin{array}{lclcrcll} \mu_{r\cdot a}(s) & = & \mu_a(s/r) & & & & \text{if} & r \neq 0, \\ \mu_{0\cdot a}(0) & = & 1, & \mu_{0\cdot a}(s) & = & 0 & \text{if} & s \neq 0, \end{array} \tag{2}$$

is called the *product* of r and a. The properties of sum (1) and product (2) are summarized in [10, 11].

Effective economic or optimization models, including the market equilibrium, demand the good handling of ordering relation on the set of numerically represented outputs. In the fuzzified models, it means to define an ordering relation over the set of fuzzy quantities. There exist numerous definitions of such relation (some of them are recollected in [11]) in the literature. Here, we use the one which is based on the paradigm that relation between vague (i. e., fuzzy) quantities is to be fuzzy, as well. The *fuzzy ordering relation* $\succeq$ used in the following sections is represented by a fuzzy subset of $\mathcal{F}(R) \times \mathcal{F}(R)$ with membership function $\nu_{\succeq}(\cdot,\cdot)$. For every pair of fuzzy quantities a, b with μ_a, μ_b, the value $\nu_{\succeq}(a,b)$ represents the possibility that $a \succeq b$, and

$$\nu_{\succeq}(a,b) = \sup\left[\min(\mu_a(r),\, \mu_b(s)) : r,\, s \in R,\, r \geq s\right]. \tag{3}$$

The above concepts of the theory of fuzzy quantities are sufficient for the presentation of the fuzzy market model suggested below.

2.2 Deterministic Cooperative Game

The cooperative game itself is not adequate for the representative description of the structure of market activities but it is closely related to the market model and the game theoretical concepts represent a pattern for some analogous components of the market (cf., [5, 6, 17, 16] and also [7, 8], e. g.).

Let us denote by I the (non-empty and finite) set of *players*. To simplify some notations, we "name" the players by natural numbers, hence $I = \{1, 2, \dots, n\}$. Every subset of I is called a *coalition*. By $\boldsymbol{K}$ we denote the set of all coalitions. A mapping $v : \boldsymbol{K} \to R$ such that $v(\emptyset) = 0$ for the empty coalition, is called *characteristic function* of the considered game. For every coalition $K \in \boldsymbol{K}$ the value $v(K)$ represents its expected total output.

The cooperative *game with transferable utility* (briefly *TU-game*) is represented by the pair

$$(I, v).$$

The TU-game (I, v) is said to be *superadditive* if for every pair of disjoint coalitions $K,\ L \subset I$

$$v(K \cup L) \geq v(K) + v(L). \tag{4}$$

The concept of TU-game is based on the idea that every realized coalition K expects to win a pay-off $v(K)$ which is distributed among its members. Such distribution is described by a real-valued vector $\boldsymbol{r}_K = (r_i)_{i \in K}$. For the coalition of all players I, every vector $\boldsymbol{r} = (r_i)_{i \in I}$ is called an *imputation*. We say that an imputation $\boldsymbol{r}$ is accessible for I if

$$\sum\nolimits_{i \in I} r_i \leq v(I),$$

and we say that it is blocked by a coalition $K \subset I$ iff

$$\sum\nolimits_{i \in K} r_i < v(K).$$

The set C of all imputations which are accessible for I and are not blocked by any coalition K, i. e.,

$$C = \left\{ \boldsymbol{r} \in R^n : \sum\nolimits_{i \in I} r_i \leq v(I),\ \forall\, K \in \boldsymbol{K}, \sum\nolimits_{i \in K}, r_i \geq v(K) \right\}, \tag{5}$$

is called a *core* of the game (I, v).

2.3 Competitive Deterministic Market

The basic market model the fuzzification of which will be investigated in further sections is defined as follows.

Even in this case we denote by I the finite and non-empty set of players; in the market model, they are usually called *agents*. We suppose that there exist m sorts of goods which are somehow distributed among agents. By the symbol x^i_j we denote the amount of the goods $j \in \{1, \dots, m\}$ owned by agent $i \in I$. We suppose that $x^i_j \geq 0$. Values x^i_j form a *distribution matrix* $\boldsymbol{x}$ with columns $\boldsymbol{x}^i = (x^i_j)_{i=1,\dots,m}$, where column $\boldsymbol{x}^i$ characterizes the *structure of property* owned by agent $i \in I$. There exists a special distribution matrix, let us denote it $\boldsymbol{a}$ with elements a^i_j and columns $\boldsymbol{a}^i$, $i \in I,\ j = 1, \dots, m$, which is called *initial distribution matrix*, and which represents the distribution of goods at the very beginning of the bargaining and exchange process. It is useful to denote the set of all distribution matrices achievable in the considered market by means of re-distribution of $\boldsymbol{a}$ by $\boldsymbol{X}$, i. e.,

$$\boldsymbol{X} = \left\{ \boldsymbol{x} = (\boldsymbol{x}^i)_{i \in I} : \forall\, i \in I, \boldsymbol{x}^i \in R^m_+, \forall\, j = 1, \dots, m, \sum\nolimits_{i \in I} x^i_j \leq \sum\nolimits_{i \in I} a^i_j \right\}. \tag{6}$$

To simplify some notations, we denote for every coalition $K \subset I$

$$\boldsymbol{X}^K = \left\{ \boldsymbol{x} \in \boldsymbol{X} : \forall\, 1, \dots, m, \sum\nolimits_{i \in K} x^i_j \leq \sum\nolimits_{i \in K} a^i_j \right\} \tag{7}$$

as the set of all distribution matrices which are accessible by re-distribution of goods inside the coalition K.

Remark 1. It is evident that we may put without loss of validity of the definitoric formula (7) $\boldsymbol{X}^{\emptyset} = \boldsymbol{X}$ for empty coalition $\emptyset$.

Remark 2. It is always evident that $\boldsymbol{X}^I = \boldsymbol{X}$ and for one-agent coalition $\{i\}$, $i \in I$,

$$\boldsymbol{X}^{\{i\}} = \left\{ \boldsymbol{x} \in \boldsymbol{X} : \forall\, j = 1, \ldots, m,\ x^i_j \le a^i_j \right\}.$$

Finally, we admit that every agent $i \in I$ evaluates the achieved distribution matrix $\boldsymbol{x}$ by a *utility* function $u_i : \boldsymbol{X} \to R$ which depends exclusively on the vector of goods $\boldsymbol{x}^i$, i. e., for any $\boldsymbol{x}$, $\boldsymbol{y} \in \boldsymbol{X}$

$$u_i(\boldsymbol{x}) = u_i(\boldsymbol{y}) \quad \text{if } \boldsymbol{x}^i = \boldsymbol{y}^i \tag{8}$$

and is non-decreasing and concave. It is natural to suppose that $u_i(\boldsymbol{x}) = 0$ if $x^i_j = 0$ for all $j = 1, \ldots, m$.

Then we call the ordered quadruple

$$\boldsymbol{M} = (I, m, \boldsymbol{a}, (u_i)_{i \in I}) \tag{9}$$

a *free exchange market*.

The exchange of goods in a market respects the prices and, vice-versa, relation between demand and supply influences their structure. The *prices* of goods form a real valued vector $p = (p_1, p_2, \ldots, p_m)$ where $p_j > 0$ for any $j = 1, \ldots, m$, and, of course, p_j is the price of good j. The set of admissible prices will be denoted by P (let us note that the notion of "admissibility" is sometimes quite significant, some price regulations do exist in numerous real economies). The prices $p \in P$ are supposed to be row vectors so that the product $p \cdot \boldsymbol{x}^i$ has sense and its result is a scalar.

For every agent $i \in I$ and price vector $p \in P$ we denote by $\mathcal{B}^i(p)$ the set of distribution matrices

$$\mathcal{B}^i(p) = \left\{ \boldsymbol{x} \in \boldsymbol{X} : p \cdot \boldsymbol{x}^i \le p \cdot \boldsymbol{a}^i \right\}, \tag{10}$$

which is called the *budget set* of agent i.

In the next sections, we call a pair $(\boldsymbol{x}, p)$, where $\boldsymbol{x} \in \boldsymbol{X}$ and $p \in P$ the *state of the market* $\boldsymbol{M}$. Some states of market, respecting the balance between demand and supply, deserve our special attention. A state of market

$$(\boldsymbol{x}, p) \in \boldsymbol{X} \times P$$

is called a *competitive equilibrium* iff for every agent $i \in I$

$$\boldsymbol{x} \in \mathcal{B}^i(p), \tag{11}$$

$$u_i(\boldsymbol{x}) \ge u_i(\boldsymbol{y}) \quad \text{for every } \boldsymbol{y} \in \mathcal{B}^i(p). \tag{12}$$

Each market $\boldsymbol{M}$ is connected with a cooperative TU-game (I, v), where the characteristic function v is defined by

$$v(K) = \max \left\{ \textstyle\sum_{i \in K} u_i(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}^K \right\}. \tag{13}$$

The pair (I, v) defined by (13) is called *market game* of the market $\boldsymbol{M}$. The most important results of the market equilibrium theory specify the relation between the core of the market game and the vector of utilities

$$(u_i(\boldsymbol{x}))_{i \in I}$$

where $(\boldsymbol{x}, p)$ is the competitive equilibrium of $\boldsymbol{M}$. Namely the fact that $(u_i(\boldsymbol{x}))_{i \in I} \in C$ under easily fulfilled assumptions.

2.4 Coalitional Competitive Market

The classical model of market and its equilibrium briefly recollected in 2.3, was rather extended in [7, 8] and several related papers. The cooperative extension is based on the idea that there exist two qualitatively different levels of cooperation. One of them represents the strictly market relations – the agents exchange the goods in order to maximize their subjective utility, respecting the existing prices. But some groups of agents reflect closed relations among them. They act as one compact participant of the market aiming to maximize the total group utility (as the sum of individual utilities of its members). That utility, connected with some total coalitional ownership of goods, is distributed among the agents forming the group in order to achieve the maximal sum of individual utilities. Even if the "external" exchange of common goods of the group respects the prices, its "internal" re-distribution does not. Nevertheless, in spite of this "collectivism" the motivation of each agent is individual – he aims to maximize his own profit under the limitations given by the compromise group agreement maximizing the sum of profits. This structure of the market demands the existence of a very specific good, called "*money*" and intermediating the re-distribution of utility among agents. In this simplified model, we do not use separate denotation for this specific good but it is useful to register its hidden existence (more attention is paid to money, e. g., in [16, 17, 7, 8]). Such groups of close and in same sense "altruistic" cooperation do exist. They are formed, e. g., by families, economic concerns or cooperatives. The framework idea of the "close groups" can be used also for modelling the market behaviour of an agent (economic subject) which diversificates his property in several separated blocks treated in the market as independent distributions of goods (such approach remembers of the parallel participation of a player in different coalitions dealt, e. g., in [1, 2, 4, 13, 14]).

For every coalition of agents $K \subset I$ we denote the *coalitional utility function* $u_K : \boldsymbol{X} \to R$ by

$$u_K(\boldsymbol{x}) = \sum\nolimits_{i\in K} u_i(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{X} \tag{14}$$

and for the vector of prices $p \in P$ also the *coalitional budget set* $\mathcal{B}^K(p)$ by

$$\mathcal{B}^K_p = \left\{\boldsymbol{x} \in \boldsymbol{X} : \sum\nolimits_{i\in K} p \cdot \boldsymbol{x}^i \le \sum\nolimits_{i\in K} p \cdot \boldsymbol{a}^i\right\}. \tag{15}$$

Let us consider a class of coalitions $\mathcal{M} \subset 2^I$ which covers the set I; in formulas

$$\bigcup\nolimits_{K\in\mathcal{M}} K = I.$$

If $(\boldsymbol{x}, p) \in \boldsymbol{X} \times P$ is a state of market $\boldsymbol{M}$ then we say that it is a *cooperative $\mathcal{M}$-equilibrium* iff for all $K \in \mathcal{M}$

$$\boldsymbol{x} \in \mathcal{B}^K(p) \tag{16}$$

$$u_K(\boldsymbol{x}) \ge u_K(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in \mathcal{B}^K(p). \tag{17}$$

The relations between competitive equilibrium (11), (12) and cooperative $\mathcal{M}$-equilibrium (16), (17) are dealt in the referred works [7, 8] and several other papers.

Of course, it is possible to analyze the relation between such cooperative market and corresponding market game. The market game defined by (13), i. e.

$$v(K) = \max\left\{u_K(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}^K\right\}, \quad K \subset I,$$

includes the cooperative behaviour of agents and it can be easily used even in this case. The specific structure of cooperative $\mathcal{M}$-equilibrium can be reflected by an analogous modification of core. Namely, the set of real-valued vectors $C_{\mathcal{M}}$, defined by

(18) $$C^{\mathcal{M}} = \left\{ \boldsymbol{r} = (r_i)_{i \in I} : \sum_{i \in I} \leq v(I), \text{ for all } K \in \mathcal{M} \ \sum_{i \in K} r_i \geq v(K) \right\}$$

is called the $\mathcal{M}$-*core* of the game (I, v). Relation between cooperative $\mathcal{M}$-equilibrium of a market $\boldsymbol{M}$ and the $\mathcal{M}$-core of corresponding market game is dealt in [7, 8], and it is consistent with the relations valid for the not cooperative case.

3 Fuzzy Utilities and Prices

The market model (9), both modifications of its equilibria, (11), (12) and (16), (17), as well as the corresponding market game (13) can be fuzzified. This fuzzification is natural – the reality of the market is closely connected with subjectivity of stand-points, vagueness of information and approximity (eventually other deformation) of data, which are typical for real economic situations.

In this chapter, we are interested in the fuzzy modifications of two components of the market which have quantitative character, namely of the utilities and prices. Let us note that utilities represent the preferences, which means essentially qualitative element of the model, but they are described by a quantitative scale of utilities. Both of the fuzzified components, the utilities and the prices, are modelled by fuzzy quantities. It is also rational to admit that other components of the model – the set of agents, the number of goods and their initial distribution – are usually well known and their fuzzification would not be adequate to the analyzed situation.

With regard to the analogy with coalitional games, it is interesting to mention, at least briefly, another possibility of fuzzification, namely the existence of fuzzy coalitions as fuzzy subsets of I. In the reality of market (or cooperative game) they reflect the possibility that the individual agents may distribute their participation among several coalitions. This may be quite possible but it is not considered in this section.

The fuzzification dealt here aims to extend the above mentioned components of the market (9)

$$\boldsymbol{M} = (I, m, \boldsymbol{a}, (u_i)_{i \in I}),$$

with the set P described in Subsection 2.3.

3.1 Fuzzy Competitive Market

Let us consider fuzzy functions $u_i^F : \boldsymbol{X} \to \mathcal{F}(R)$, $i \in I$, such that for every $\boldsymbol{x} \in \boldsymbol{X}$ the value $u_i^F(\boldsymbol{x})$ is a fuzzy quantity with membership function $\mu_{i,\boldsymbol{x}} : R \to [0, 1]$. Let, $\mu_{i,\boldsymbol{x}}(u_i(\boldsymbol{x})) = 1$ for any $i \in I$, $\boldsymbol{x} \in \boldsymbol{X}$, Analogously to (8) we suppose that

(19) $$\mu_{u,\boldsymbol{x}}(r) = \mu_{i,\boldsymbol{y}}(r) \quad \text{for all } r \in R \text{ if } \boldsymbol{x}^i = \boldsymbol{y}^i.$$

Moreover, if $x_j^i = 0$ for all $j = 1, 2, \ldots, m$ then

$$\mu_{i,\boldsymbol{x}}(0) = 1, \quad \mu_{u,\boldsymbol{x}}(r) = 0 \quad r \in R, \ r \neq 0.$$

Then we call the mappings u_i^F, $i \in I$, *fuzzy utility functions* and their values $u_i^F(\boldsymbol{x})$ *fuzzy utilities.*

Remark 3. Fuzzy utilities u_i^F are fuzzy extensions of the utilities u_i, $i \in I$, in (9) in the sense that if $\mu_{i,\boldsymbol{x}}(u_i(\boldsymbol{x})) = 1$ and $\mu_{i,\boldsymbol{x}}(r) = 0$ for $r \neq u_i(\boldsymbol{x})$ then u_i^F fulfil the properties of the crisp utility functions formulated in Subsection 2.3.

Let us consider for every $j = 1, \dots, m$ a fuzzy quantity q_j with membership function $\pi_j : R \to [0, 1]$ such that $\pi_j(p_j) = 1$ for some $(p_j)_{j=1,\dots,m} \in P$, and

$$\pi_j(r) = 0 \quad \text{for } r \leq 0.$$

Then fuzzy quantities q_j are called *fuzzy prices* and the vector $(q_j)_{j=1,\dots,m} = q$ is called *fuzzy price vector*.

Remark 4. The fuzzy prices q_j are fuzzy extensions of the crisp prices p_j in the sense that if the fuzzy quantities are reduced into single possible value, i.e., $\pi_j(r) = 0$ for $r \neq p_j$, then the vector q has the properties of the deterministic price vector $p \in P$ dealt in Subsection 2.3.

Let us denote the set of all fuzzy price vectors by P^F.

The quadruple

$$\boldsymbol{M}^F = \Big(I, m, \boldsymbol{a}, (u_i^F)_{i \in I}\Big) \quad \text{and the set } P^F \tag{20}$$

is called *fuzzy competitive market* extending the deterministic market $\boldsymbol{M}$.

The fuzziness of some components of the model means that some of the concepts derived from it are fuzzy, as well. This consequent fuzziness will be the main topic of our analysis in the remaining part of this subsection.

For every agent $i \in I$, every structure of his property $\boldsymbol{x}^i$ (which is the relevant column of the distribution matrix $\boldsymbol{x} \in \boldsymbol{X}$), and for any vector of fuzzy prices $q \in P^F$ we may easily operate with the scalar product

$$q \cdot \boldsymbol{x}^i = q_1 \cdot x_1^i \oplus q_2 \cdot x_2^i \oplus \cdots \oplus q_m \cdot x_m^i, \tag{21}$$

where each product on the right-hand-side of (21) is defined by (2) and the sums of fuzzy quantities in (21) are defined by (1). Hence, formula (21) defines a fuzzy quantity. The same is correct for the scalar product $q \cdot \boldsymbol{a}^i$ where $\boldsymbol{a}^i$ is the relevant column of the initial distribution matrix. Let us note that, using (3), we are able to compare these two fuzzy quantities and that the value $\nu_{\succeq}\ (q \cdot \boldsymbol{a}^i,\ q \cdot \boldsymbol{x}^i)$ of the membership function $\nu_{\succeq}$ specifies the possibility that the fuzzy ordering relation

$$q \cdot \boldsymbol{a}^i \succeq q \cdot \boldsymbol{x}^i$$

is valid.

The above operations justify the definition of the fuzzy subset ${}^F\mathcal{B}^i(q)$ of $\boldsymbol{X}$ with membership function $\beta_{i,q} : \boldsymbol{X} \to [0, 1]$ defined by

$$\beta_{i,q}(\boldsymbol{x}) = \nu_{\succeq}(q \cdot \boldsymbol{a}^i,\ q \cdot \boldsymbol{x}^i), \tag{22}$$

called *fuzzy budget set* of agent i and fuzzy prices q. It is easy to verify that the concept of fuzzy budget set is a fuzzy extension of the deterministic budget set $\mathcal{B}^i(p)$ defined by (10), as follows from the next statement.

Remark 5. If q, $q' \in P^F$ and π_j, π'_j, $j = 1, \ldots, m$, are corresponding membership functions such that for any $r \in R$, $\pi_j(r) \geq \pi'_j(r)$ for all $j = 1, \ldots, m$, then ${}^F\!\mathcal{B}^i(q) \supset^F \mathcal{B}^i(q')$ in the fuzzy set theoretical sense, i. e. $\beta_{i,q}(\boldsymbol{x}) \geq \beta_{i,q'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Analogously to the deterministic market model we call the pair $(\boldsymbol{x}, q) \in \boldsymbol{X} \times P^F$ a *state of the fuzzy market* $\boldsymbol{M}^F$.

The construction of the above concepts is motivated by the endeavour to introduce the concept of market equilibrium adequate to the considered type of market. It is intuitively evident that the equilibrium of fuzzy market is to be a vague, i. e. fuzzy, concept. It means that such equilibria form a fuzzy subset of the Cartesian product $\boldsymbol{X} \times P^F$. We denote its membership function by $\rho : \boldsymbol{X} \times P^F \to [0, 1]$ and its value $\rho(\boldsymbol{x}, q)$, denoting the possibility that the state of fuzzy market $(\boldsymbol{x}, q)$ is an equilibrium, is defined by

$$\rho(\boldsymbol{x}, q) = \min\left[\beta(\boldsymbol{x}, q),\ \delta(\boldsymbol{x}, q)\right], \tag{23}$$

where

$$\beta(\boldsymbol{x}, q) = \min\left(\beta_{i,q}(\boldsymbol{x}) : i \in I\right) \tag{24}$$

denotes the possibility that $\boldsymbol{x}$ belongs to all fuzzy budget sets ${}^F\!\mathcal{B}^i(q)$, see (22), and

$$\delta(\boldsymbol{x}, q) = \min\left[\beta(\boldsymbol{y}, q), \min\left(\nu_{\succeq}(u_i^F(\boldsymbol{x}),\ u_i^F(\boldsymbol{y})) : i \in I\right) : \boldsymbol{y} \in \boldsymbol{X}\right], \tag{25}$$

denotes the possibility that the fuzzy utility of $\boldsymbol{x}$ is greater than the fuzzy utility of $\boldsymbol{y}$ for any $\boldsymbol{y}$ which may belong to the fuzzy budget sets ${}^F\!\mathcal{B}^i(q)$ for all agents $i \in I$. The fuzzy subset of $\boldsymbol{X} \times P^F$ with membership function ρ is called *fuzzy competitive equilibrium* of $\boldsymbol{M}^F$.

It is not difficult to conclude from the above definitions that the fuzzy equilibrium extends the deterministic concept of equilibrium (described in Subsection 2.3) and that the increasing fuzziness of the input components increases the fuzziness of equilibria. This heuristic conclusion can be formulated in the following statements.

Lemma 1. If q, $q' \in P^F$ are two fuzzy prices vectors with π_j, π'_j, $j = 1, \ldots, m$, respectively, if $\pi_j(r) \geq \pi'_j(r)$ for all $r \in R$ then $\rho(\boldsymbol{x}, q) \geq \rho(\boldsymbol{x}, q')$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

Proof. The statement follows from (24) and (25). If $\pi_j(r) \geq \pi'_j(r)$ for all $r \in R$ and $j = 1, \ldots, m$ and if we denote by $\pi_{j,\boldsymbol{x}}$ and $\pi'_{j,\boldsymbol{x}}$ the membership functions of fuzzy quantities $q_j \cdot x_j^i$ and $q'_j \cdot x_j^i$ for arbitrary $i \in I$. Then (2) implies that $\pi_{j,\boldsymbol{x}}(r) \geq \pi'_{j,\boldsymbol{x}}(r)$ for any $r \in R$. Hence, due to (3) and (22), $\beta_{i,q}(\boldsymbol{x}) \geq \beta_{j,q'}(\boldsymbol{x})$ and (24), together with (23), implies the statement. □

Lemma 2. Let us consider $u_i^F, \overline{u}_i^F$ for some $i \in I$, with membership functions $\mu_{i,\boldsymbol{x}}$, $\overline{\mu}_{i,\boldsymbol{x}}$, $\boldsymbol{x} \in \boldsymbol{X}$, such that for all $r \in R$

$$\mu_{1,\boldsymbol{x}}(r) \geq \overline{\mu}_{i,\boldsymbol{x}}(r).$$

Let us denote

$$\boldsymbol{M}^F = \left(I, m, \boldsymbol{a}, \left(u_i^F\right)_{i \in I}\right), \quad \overline{\boldsymbol{M}}^F = \left(I, m, \boldsymbol{a}, \left(\overline{u}_i^F\right)_{i \in I}\right)$$

and by ρ, $\overline{\rho}$ the membership functions of fuzzy competitive equilibria of the fuzzy markets $\boldsymbol{M}^F$ and $\overline{\boldsymbol{M}}^F$, respectively for some fuzzy prices $q \in P^F$. Then

$$\rho(\boldsymbol{x}, q) \geq \overline{\rho}(\boldsymbol{x}, q).$$

P r o o f. The statement follows from (23) and related definitions, especially from (25). Under the assumptions of this lemma, mappings $\delta(\boldsymbol{x}, q)$ and $\delta(\boldsymbol{x}, \overline{q})$ fulfil the inequality

$$\delta(\boldsymbol{x}, q) \geq \delta(\boldsymbol{x}, \overline{q}),$$

which, together with (23) implies the statement. □

Theorem 1. The fuzzy market $\boldsymbol{M}^F = (I, m, \boldsymbol{a}, (u_i^F)_{i \in I})$with set of fuzzy prices P^F is an extension of the deterministic market $\boldsymbol{M} = (I, m, \boldsymbol{a}, (u_i)_{i \in I})$ with set of prices P, and fuzzy equilibria $(\boldsymbol{x}, q)$ of $\boldsymbol{M}^F$ are fuzzy extensions of deterministic equilibria $(\boldsymbol{x}, q)$ of $\boldsymbol{M}$ if q is fuzzy extension of p (i. e. $\pi_j(r) = 1$ iff $r = p_j$ for all $j = 1, \ldots, m$ and $\mu_{i,\boldsymbol{x}}(r) = 1$ iff $r = u_i(\boldsymbol{x})$).

P r o o f. The theorem follows from the previous statements, namely Lemma 1, Lemma 2, and Remarks 3, 4, 6, immediately. □

Corollary 1. The previous theorem together with Remark 5 implies that if fuzzy quantities $u_i^F(\boldsymbol{x})$ and q_j for $i \in I$, $\boldsymbol{x} \in \boldsymbol{X}$ and $j = 1, \ldots, m$ condensate into single possible values $u_i(\boldsymbol{x}), p_j$, i. e. $\mu_{i,\boldsymbol{x}}(r) = 0$ for $r \neq u_i(\boldsymbol{x})$ and $\pi_j(r) = 0$ for $r \neq p_j$, $i \in I$, $\boldsymbol{x} \in \boldsymbol{X}$ and $j = 1, \ldots, m$, then the market $\boldsymbol{M}^F$ is identical with $\boldsymbol{M}$ and fuzzy equilibrium $(\boldsymbol{x}, q)$ condensates into crisp equilibrium $(\boldsymbol{x}, p)$ in the sense that $\rho(\boldsymbol{y}, q') = 0$ for $\boldsymbol{y} \neq \boldsymbol{x}$ or $q' \neq q$.

Theorem 2. The fuzzy competitive equilibrium $(\boldsymbol{x}, q) \in \boldsymbol{X} \times P^F$ can be transformed into a fuzzy subset of $\boldsymbol{X} \times P$, i. e., to a fuzzy set of deterministic equilibria.

P r o o f. Due to the above definitions, every fuzzy equilibrium is a fuzzy subset of $\boldsymbol{X} \times P^F$ with membership function ρ having values $\rho(\boldsymbol{x}, q)$. Distribution matrices are crisp objects but each $q \in P^F$ is a vector of fuzzy subsets of R_+ with memberships π_j. Let us define a membership function $\pi : P \to [0, 1]$ as

$$\pi(p) = \min\left(\pi_j(p_j) : j = 1, 2, \ldots, m\right).$$

Then it is possible to define a fuzzy subset of $\boldsymbol{X} \times P$ with membership function $\rho^* : \boldsymbol{X} \times P \to [0, 1]$, where for $\boldsymbol{x} \in \boldsymbol{X}$ and $p \in P$

$$\rho^*(\boldsymbol{x}, p) = \min\left(\rho(\boldsymbol{x}, q),\ \pi(p) : p \in P\right).$$

□

3.2 Fuzzy Market Game

It is possible to proceed analogously to the deterministic model and to derive a coalitional TU-game in some sense connected with the fuzzy market $\boldsymbol{M}^F$. It means that it is necessary to define its characteristic function, by means of a procedure modifying (13) for the environment of fuzzy utilities. First of all, we define for any coalition $K \subset I$, $K \neq \emptyset$, and any $\boldsymbol{x} \in \boldsymbol{X}$ the fuzzy quantity $u_K^F(\boldsymbol{x})$ by

$$u_K^F(\boldsymbol{x}) = \sum\nolimits_{i\in K}^{\oplus} u_i^F(\boldsymbol{x}), \tag{26}$$

where $\sum^{\oplus}$ means the fuzzy sum using the operation $\oplus$ (see (1)). More precisely, if $K = \{i_1, i_2, \ldots, i_k\}$ then

$$u_K^F(\boldsymbol{x}) = u_{i_1}^F(\boldsymbol{x}) \oplus u_{i_2}^F(\boldsymbol{x}) \oplus \cdots \oplus u_{i_k}^F(\boldsymbol{x}).$$

By $\mu_{K,\boldsymbol{x}} : R \to [0,1]$ we denote the membership function of $u_K^F(\boldsymbol{x})$. Due to [10], $u_K(\boldsymbol{x})$ is a fuzzy quantity, as well. Then we may, using (3), construct for every $K \subset I$, $K \neq \emptyset$, the maximum of fuzzy quantities $u_K^F(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{X}^K$ (see (7)) as a fuzzy quantity $w(K)$ by the following procedure. For every $\boldsymbol{x} \in \boldsymbol{X}^K$ we find

$$\min\left(\nu_{\succeq}\left(u_K^F(\boldsymbol{x}),\, u_K^F(\boldsymbol{y})\right) : \boldsymbol{y} \in \boldsymbol{X}^K\right) \tag{27}$$

as the possibility that $u_K^F(\boldsymbol{x}) \succeq u_K^F(\boldsymbol{y})$ for all $\boldsymbol{y} \in \boldsymbol{X}^K$, and then put

$$w(K) = u_K^F(\boldsymbol{x}) \tag{28}$$

for the $\boldsymbol{x} \in \boldsymbol{X}^K$ for which the value (27) is maximal. Let us note that the closedness of $\boldsymbol{X}^K$ following from (7) implies the correctness of the above maxima and minima. In the following text, we denote the membership function of $w(K)$ by χ_K. For empty coalition $\emptyset$ we put $\chi_\emptyset(0) = 1$, $\chi_\emptyset(r) = 0$ for $r \in R$, $r \neq 0$. In the terms of [12], it is easy and natural to interpret the pair

$$(I, w)$$

as a cooperative game with transferable utility and with fuzzy pay-offs. This game will be called *fuzzy market game*.

The theory of TU-games with fuzzy pay-offs is relatively new. It is developed since the nintieths and the elementary or basic concepts and results are summarized in [12]. The model is further investigated and some of its modifications are suggested (cf. [15]).

Theorem 3. *Let $\boldsymbol{M}$ be a competitive market and $\boldsymbol{M}^F$ be its fuzzy extension in the above sense. If (I, v) is the market game of $\boldsymbol{M}$ and (I, w) is the fuzzy market game of $\boldsymbol{M}^F$ then w is a fuzzy extension of v, i.e. $\chi_K(v(K)) = 1$ for any coalition $K \subset I$.*

Proof. The assumption that $\boldsymbol{M}^F$ is fuzzy extension of $\boldsymbol{M}$ means that

$$\mu_{i,\boldsymbol{x}}(u_i(\boldsymbol{x})) = 1 \tag{29}$$

for all $i \in I$, $\boldsymbol{x} \in \boldsymbol{X}^K$. It means that if $\overline{\boldsymbol{x}} \in \boldsymbol{X}^K$ is the distribution matrix for which $u_K(\overline{\boldsymbol{x}}) \geq u_k(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{X}^K$. Then (29) means that $\nu_{\succeq}(u_K^F(\overline{\boldsymbol{x}}), u_k^F(\boldsymbol{y})) = 1$ for all $\boldsymbol{y} \in \boldsymbol{X}^K$, and, consequently,

$$v(K) = u_K(\overline{\boldsymbol{x}}), \quad \chi_K(v(K)) = 1. \qquad \square$$

Corollary 2. If $\boldsymbol{M}^F$ is a fuzzy market the fuzziness of which is condensed into a competitive market $\boldsymbol{M}$, i. e., $\mu_{i,\boldsymbol{x}}(u_i(\boldsymbol{x})) = 1$, $\mu_{i,\boldsymbol{x}}(r) = 0$ if $r \neq u_i(\boldsymbol{x})$, for all $i \in I$, $\boldsymbol{x} \in \boldsymbol{X}$ then the fuzziness of the fuzzy market game (I, w) of $\boldsymbol{M}^F$ is condensed into the market $\chi_K(v(K)) = 1$, $\chi_K(r) = 0$ if $r \neq v(K)$, for all $K \subset I$, as follows from Theorem 2 and previous definitions.

The relation between fuzzy competitive market and its fuzzy market game represents an inspirative topic for detailed research, most of which is to be done, yet. It is unavoidable to respect the fact that TU-games with fuzzy characteristic function (I, w) do not fully copy some of the useful properties of the deterministic TU-games (I, v), however desirable it could be. This discrepancy follows from the algebraic properties of fuzzy quantities which are not identical with the properties of crisp real or integer numbers. Significant differences are connected with the notions of fuzzy zero and opposite element which are selfevident for groups of deterministic numbers. These differences cause some essential complications, especially, regarding the relations between convexity of characteristic function and existence of core, as well as relations between superadditivity and convexity. Deeper analysis of these consequences of all these specific features of fuzziness in TU-games can be found in [12]. We may note that the relation between fuzzy competitive equilibrium and core (it means fuzzy core) of the fuzzy market game is not simple and that it cannot be a simple analogy of its deterministic counterpart.

On the other hand, the situation in a fuzzy market $\boldsymbol{M}^F$ with fuzzy prices P^F and its fuzzy market game (I, w) becomes more lucid if we accept the fact that many concepts which are in the deterministic case strictly limited are in the fuzzy case related (in various degree of possibility) to all relevant objects. It regards, e. g., the prices – a fuzzy price $q \in P^F$ may represent, with possibility $\pi(p)$ any crisp price $p \in P$. Similarly, fuzzy equilibrium $(\overline{\boldsymbol{x}}, \overline{q})$ represents a fuzzy subset of $\boldsymbol{X} \times P^F$, it means any state of fuzzy market $(\boldsymbol{x}, q)$ with possibility $\rho(\boldsymbol{x}, q)$. Together with the fuzziness of q it means that fuzzy equilibrium of $\boldsymbol{M}^F$ can be interpreted as fuzzy set of crisp equilibria in $\boldsymbol{M}$ (see Theorem 2).

Using the concepts and results summarized in [12], we analyze at least the basic properties of the fuzzy market game (I, w) where for every $K \subset I$, $w(K)$ is a fuzzy quantity with modal value $v(K)$. In the first step we verify the superadditivity. In TU-games with fuzzy pay-offs of coalitions, the superadditivity is a fuzzy property, as well. It means that if we denote by Γ_I the set of all coalitional games with fuzzy characteristic functions, then the *fuzzy superadditive* games form a fuzzy subset of Γ_I with membership function $\sigma : \Gamma_I \to [0, 1]$. For every game (I, w) the value $\sigma(w)$ is defined by

$$\sigma(w) = \min\left[\nu_{\succeq}(w(K \cup L),\, w(K) \oplus w(L)) : K,\, L \subset I,\, K \cap L = \emptyset\right] \tag{30}$$

where definition (3) was used, and $\sigma(w)$ determines the possibility that the game (I, w) is superadditive.

Lemma 3. If $\boldsymbol{M}$ is a deterministic competitive market and (I, v) its market game then (I, v) is superadditive.

P r o o f . Let us consider a market (9) and its game (13). Let us consider disjoint K, $L \subset I$ and sets $\boldsymbol{X}^K$, $\boldsymbol{X}^L$, $\boldsymbol{X}^{K \cup L}$, defined by (7). Then it is easy to see that $\boldsymbol{X}^{K \cup L} \supset \boldsymbol{X}^K \cap \boldsymbol{X}^L$.

Moreover, as for all $i \in I$ $u_i(\boldsymbol{x}) = u_i(\boldsymbol{y})$ if $\boldsymbol{x}^{i=}\boldsymbol{y}^i$ (see Subsection 2.3) then also $u_K(\boldsymbol{x}) = u_K(\boldsymbol{y})$ if $\boldsymbol{x}^i = \boldsymbol{y}^i$ for all $i \in K$ and the same is valid for L. It means that

$$\begin{aligned} v(K \cup L) &= \max\left(\sum\nolimits_{i \in K \cup L} u_i(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}^{K \cup L}\right) \geq \\ &\geq \max\left(\sum\nolimits_{i \in K} u_i(\boldsymbol{x}) + \sum\nolimits_{i \in L} u_i(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}^K \cap \boldsymbol{X}^L\right) = \\ &= \max\left(\sum\nolimits_{i \in K} u_i(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}^K\right) + \max\left(\sum\nolimits_{i \in L} u_i(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X}^L\right) = \\ &= v(K) + v(L). \end{aligned}$$

□

Theorem 4. Let fuzzy competitive market $\boldsymbol{M}^F$ be a fuzzy extension of deterministic market $\boldsymbol{M}$, and let (I, w) be its fuzzy market game. Then (I, w) is certainly fuzzy superadditive, i. e. $\sigma(w) = 1$.

Proof. If (I, v) is the market game of $\boldsymbol{M}$ then, due to Theorem 3, (I, w) is fuzzy extension of (I, v). It is easy to verify (the formal statement is presented, e. g., in [12]) that the superadditivity of (I, v) implies the fuzzy superadditivity of (I, w) with maximal possibility. It means that $\sigma(w) = 1$. □

Corollary 3. Fuzzy superadditivity of (I, w) derived from $\boldsymbol{M}^F$ is a fuzzy extension of the deterministic superadditivity of (I, v) derived from $\boldsymbol{M}$, if $\boldsymbol{M}^F$ is fuzzy competitive market extending $\boldsymbol{M}$.

Let us consider a set of coalitions $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$. If $K_j \cap K_\ell = \emptyset$ for $j \neq \ell$, $j, \ell = 1, 2, \ldots, m$, and if $K \cup K_2 \cup \cdots \cup K_m = I$ then we say that $\mathcal{K}$ is a *coalitional structure*.

Remark 7. If (I, w) is a fuzzy market game of a fuzzy market and if $\mathcal{K} = \{K_1, \ldots, K_m\}$ is a coalitional structure then certainly

$$w(I) \succeq w(K_1) \oplus \cdots \oplus w(K_m),$$

i. e., $\nu_{\succeq}(w(I),\, w(K_1) \oplus \cdots \oplus w(K_m)) = 1$, as follows from Theorem 4, immediately.

A very important result, probably the crucial one, of the theory of deterministic exchange market equilibria, regards their relation to the core of respective market game (see, e. g., [5, 16, 17], and also [6] or [7, 8]). Analogous relation for fuzzy markets is not obvious and it deserves deeper investigation in the future. Nevertheless, here it is desirable to formulate at least the concept of core of fuzzy market game and its basic properties. The results presented below are based on the general properties of cooperative TU-games with fuzzy pay-offs summarized in [12].

Let us start with the concept of fuzzy core. In this work, we respect the methodological paradigm due to which the properties and elements derived from the fuzzified TU-game model are to be fuzzy. Related to the concept of core, this principle means that the core is to be defined as fuzzy subset of imputations. Analogously to Subsection 2.2, we suppose that $I = \{1, 2, \ldots, n\}$ and by imputation we call any real-valued vector $\boldsymbol{r} = (r_1, r_2, \ldots, r_n) \in R^n$. Analogously to (5), we characterize a core as a set of imputations which are accessible for the maximal coalition I, and which cannot be blocked by any coalition $K \subset I$ (including $K = I$). The accessibility and blocking are defined by means of relations between the components of the considered imputations, and the values of the characteristic function for the relevant coalitions. If the values of characteristic functions are fuzzy quantities, and it is our case in this subsection, then both concepts – accessibility and blocking – are fuzzy and the inequalities in question are $\succeq$ with membership function $\nu_{\succeq} : \mathcal{F}(R) \times \mathcal{F}(R) \to [0,1]$ (cf. (3)).

First, we simplify the notation. Every real number $r \in R$ may be considered for fuzzy quantity with the possibility concentrated in a single value. To distinguish between these two interpretations of a real number, we denote the "concentrated" fuzzy quantity by $\langle r \rangle$, with membership function

$$\mu_{\langle r \rangle}(r) = 1, \quad \mu_{\langle r \rangle}(r') = 0 \quad \text{for } r \neq r'. \tag{31}$$

Let us note that, e. g., second part of (2) can be re-formulated as $0 \cdot a = \langle 0 \rangle$ for any $a \in \mathcal{F}(R)$.

Remark 8. It is easy to see that for $r \in R$, $\langle r \rangle$, $a \in \mathcal{F}(R)$, formula (3) can be simplified as

$$\begin{aligned} \nu_{\succeq}(\langle r \rangle, a) &= \max\left(\mu_a(r') : r' \geq r\right), \\ \nu_{\succeq}(a, \langle r \rangle) &= \max\left(\mu_a(r') : r \geq r'\right). \end{aligned}$$

The *fuzzy core* will be defined as a fuzzy subset of R^ndenoted C_w, with membership function $\gamma : R^n \to [0,1]$ defined for every $\boldsymbol{r} \in R^n$ by

$$\gamma(\boldsymbol{r}) = \min\left(\nu_{\succeq}\left(w(I), \left\langle \textstyle\sum_{i \in I} r_i \right\rangle\right), \widehat{\gamma}(\boldsymbol{r})\right), \tag{32}$$

where

$$\widehat{\gamma}(\boldsymbol{r}) = \min\left(\nu_{\succeq}\left(\left\langle \textstyle\sum_{i \in K} r_i \right\rangle, v(K)\right) : K \subset I\right).$$

In (32), $\nu_{\succeq}\left(w(I)\left\langle \sum_{i \in I} r_i \right\rangle\right)$ denotes the possibility that $\boldsymbol{r}$ is accessible for the coalition I, i. e.

$$w(I) \succeq \left\langle \textstyle\sum_{i \in I} r_i \right\rangle,$$

and $\widehat{\gamma}(\boldsymbol{r})$ denotes the possibility that $\boldsymbol{r}$ cannot be blocked by any coalition $K \subset I$, i. e.

$$\left\langle \textstyle\sum_{i \in K} r_i \right\rangle \succeq w(K) \quad \text{for all } K \subset I.$$

Theorem 5. Let $\boldsymbol{M}$ be a deterministic market with market game (I, v) and let $\boldsymbol{M}^F$ be its fuzzy extension with fuzzy market game (I, w), where χ_K, $K \subset I$, are membership functions of $w(K)$. Let us denote by C_v the core of (I, v) and by C_w the fuzzy core of (I, w) with membership function $\gamma : R^n \to [0,1]$. Then, $\boldsymbol{r} \in C_v$ implies $\gamma(\boldsymbol{r}) = 1$ for $\boldsymbol{r} \in R^n$.

P r o o f. Let $\boldsymbol{r} \in C_v$. Due to (5),

$$\textstyle\sum_{i \in I} r_i \leq v(I) \quad \text{and} \quad \sum_{i \in K} r_i \geq v(K)$$

for all $K \subset I$. Using (28), (26) and the assumption that

$$\mu_{K,\boldsymbol{x}}(v(K)) = 1$$

(as follows from the definition of u_i^F, $i \in I$) it is easy to see that

$$\nu_{\succeq}\left(w(I), \left\langle \sum_{i \in I} r_i \right\rangle\right) = 1$$

and

$$\nu_{\succeq}\left(\left\langle \sum_{i \in K} r_i \right\rangle, w(K)\right) = 1, \quad K \subset I,$$

which means that $\widehat{\gamma}(r) = 1$. Hence, $\gamma(\boldsymbol{r}) = 1$. □

Theorem 6. Let us preserve the notation of Theorem 5 and suppose that for all $i \in I$, $u_i^F(\boldsymbol{x}) = \langle u_i(\boldsymbol{x}) \rangle$ for each $\boldsymbol{x} \in \boldsymbol{X}$. Then for any $\boldsymbol{r} \in R^n$, $\gamma(\boldsymbol{r}) = 0$ if $\boldsymbol{r} \notin C_v$.

Proof. The assumption, together with (26) immediately mean that

$$u_K^F(\boldsymbol{x}) = \langle u_K(\boldsymbol{x}) \rangle \quad \text{for all } \boldsymbol{x} \in \boldsymbol{X}, \ K \subset I.$$

It means, due to (28), that

$$w(K) = \langle v(K) \rangle \quad \text{for all } K \subset I$$

and, consequently, $\gamma(\boldsymbol{r}) \neq 0$ iff $\boldsymbol{r} \in C_v$, as follows from the definition of fuzzy core. □

Corollary 1. Theorems 5 and 6 immediately imply that if the fuzzy quantities $u_i^F(\boldsymbol{x})$ are concentrated in a single possible value (i. e., under the assumption of Theorem 6) then $\gamma(\boldsymbol{r}) \in \{0, 1\}$ for any $\boldsymbol{r}$, more exactly, $\gamma(\boldsymbol{r}) = 1$ if $\boldsymbol{r} \in C_v$ and $\gamma(\boldsymbol{r}) = 0$ if $\boldsymbol{r} \notin C_v$.

Corollary 2. The previous Theorems 5 and 6 imply that the fuzzy core C_w of fuzzy market game (I, w) is a fuzzy extension of the deterministic core C_v of market game (I, v) if the fuzzy market $\boldsymbol{M}^F$ is a fuzzy extension of the market $\boldsymbol{M}$.

The relation between fuzzy core of fuzzy market game and the fuzzy equilibrium of the fuzzy market offers an inspirative topic for more detailed investigation. As the formal structure of fuzzy concepts is more complex than the structure of their deterministic counterparts, it is realistic to expect that even the relation between equilibria and core is in the fuzzified model more complicated and more varied than those derived for the classical deterministic market.

It is also necessary to respect certain disproportion between the mathematical structures describing the utilities, budget sets, core, characteristic function and equilibrium (which are fuzzy) and the individual imputations forming the core (which are crisp vectors even in the fuzzified model).

The type of open problems which could be solved regards the relation between fuzziness of equilibria and core. Let us consider a competitive market

$$\boldsymbol{M} = (I, m, \boldsymbol{a}, (u_i)_{i \in I})$$

with a space of prices P, and its fuzzy extension

$$\boldsymbol{M}^F = \left(I, m, \boldsymbol{a}, (u_i^F)_{i\in I}\right)$$

with space of fuzzy prices P^F. Preserving the notations used in this section, the relation between the membership functions

$$\rho(\boldsymbol{x}, q) \quad \text{and} \quad \gamma(\boldsymbol{r}), \quad \boldsymbol{x} \in \boldsymbol{X},\ q \in P^F,\ \boldsymbol{r} \in R^n$$

is to be in the focus of the eventual future investigation. The principally different structure of R^n on one side and $\boldsymbol{X} \times P^F$ on the other shows that this relation will not be direct or immediate. It is rational to expect rather the results in which this relation is intermediated by some other components of the market model.

Theorem 7. Let $(\boldsymbol{x}, p)$ be a competitive equilibrium of market $\boldsymbol{M}$. Let us denote by $\boldsymbol{u} = (u_i(\boldsymbol{x}))_{i\in I}$. Then $\gamma(\boldsymbol{u}) = 1$ in the fuzzy market $\boldsymbol{M}^F$ extending $\boldsymbol{M}$ and its fuzzy market game (I, w).

Proof. If $(\boldsymbol{x}, p)$ is an equilibrium then, due to [5, 17], $\boldsymbol{u} \in C_v$, where C_v is the core of (I, v). By Theorem 5, $\gamma(\boldsymbol{u}) = 1$ for the fuzzy core C_w of (I, w). □

3.3 Fuzzy Cooperative Market

The competitive fuzzy market concept can be generalized to its cooperative version, analogously to the procedure used in Subsection 2.4. Even its interpretation is analogous to the one given in Section 2.

Let us recollect formula (26) by which the fuzzy quantities $u_K^F(\boldsymbol{x})$ are defined for any $\boldsymbol{x} \in \boldsymbol{X}$ and $K \subset I$, $K \notin \emptyset$, as sums of the fuzzy quantities $u_i^F(\boldsymbol{x})$ for $i \in K$. The membership function of $u_K^F(\boldsymbol{x})$ will be denoted by $\mu_{K,\boldsymbol{x}} : R \to [0,1]$ and constructed by repetitive application of (1), which is correct as the operation $\oplus$ is associative on $\mathcal{F}(R)$ (see [10]).

Then it is possible to extend (15) for the environment of fuzzy market by means of the following procedure. For every non-empty coalition $K \subset I$, every player $i \in K$ and every fuzzy price vector $q = (q_j)_{i=1,\dots,m}$ we use (21) to define fuzzy quantity $q \cdot \boldsymbol{x}^i$ for each $\boldsymbol{x} = (\boldsymbol{x}^i)_{i\in I} \in \boldsymbol{X}$.

If $K = \{i_1, i_2, \dots, i_K\}$ then we denote

$$\sum\nolimits_{i\in K}^{\oplus} q\, \boldsymbol{x}^i = q \cdot \boldsymbol{x}^{i_1} \oplus q \cdot \boldsymbol{x}^{i_2} \oplus \cdots \oplus q \cdot \boldsymbol{x}^{i_K}, \tag{33}$$

and analogously for $\sum\nolimits_{i\in K}^{\oplus} q \cdot \boldsymbol{a}^i$ (cf. (26)). Having introduced the above symbols, we may define the *coalitional fuzzy budget set* ${}^F\mathcal{B}^K(q)$ for $q \in P^F$ as a fuzzy subset of $\boldsymbol{X}$ with membership function $\beta_{K,q} : \boldsymbol{X} \to [0,1]$ where

$$\beta_{K,q}(\boldsymbol{x}) = \nu_{\succeq}\left(\sum\nolimits_{i\in K}^{\oplus} q \cdot \boldsymbol{a}^i, \sum\nolimits_{i\in K}^{\oplus} q \cdot \boldsymbol{x}^i\right), \quad \boldsymbol{x} \in \boldsymbol{X}. \tag{34}$$

It means that $\beta_{K,q}(\boldsymbol{x})$ denotes the possibility that

$$\sum\nolimits_{i\in K}^{\oplus} q \cdot \boldsymbol{a}^i \succeq \sum\nolimits_{i\in K}^{\oplus} q \cdot \boldsymbol{x}^i,$$

(cf. (3), (22) and (15)).

Remark 9. If $q \in P^F$ is fuzzy extension of $p \in P$ (i. e., $\pi_j(p) = 1$) for all $j = 1, \dots, m$ then

$$\sum_{i \in K} p \cdot \boldsymbol{x}^i \quad \text{and} \quad \sum_{i \in K} p \cdot \boldsymbol{a}^i$$

are the modal values of fuzzy quantites

$$\sum_{i \in K}^{\oplus} q \cdot \boldsymbol{x}^i \quad \text{and} \quad \sum_{i \in K}^{\oplus} q \cdot \boldsymbol{a}^i,$$

respectively. It means that their values are achieved with possibility 1 (see Subsection 2.1).

Lemma 4. If $\boldsymbol{M}^F$ is fuzzy extension of $\boldsymbol{M}$ and $q \in P^F$ fuzzy extension of $p \in P$ then for any $K \subset I$, $K \neq \emptyset$, ${}^F\mathcal{B}^K(q)$ is fuzzy extension of $\mathcal{B}^K(p)$, i. e., $\beta_{K,q}(\boldsymbol{x}) = 1$ for any $\boldsymbol{x} \in \mathcal{B}^K(p)$.

P r o o f. Due to (33) and Remark 9,

$$\nu_{\succeq}\left(\sum_{i \in K}^{\oplus} q \cdot \boldsymbol{a}^i, \sum_{i \in K}^{\oplus} q \cdot \boldsymbol{x}^i\right) = 1$$

if

$$\sum_{i \in K} p \cdot \boldsymbol{a}^i \geq \sum_{i \in K} p \cdot \boldsymbol{x}^i.$$

The validity of the statement follows from this implication. □

Lemma 5. Under the assumptions of the previous Lemma 4, if $\boldsymbol{x} \notin \mathcal{B}^K(p)$ then $\beta_{K,\boldsymbol{x}}(q) < 1$.

P r o o f. If $\boldsymbol{x} \notin \mathcal{B}^K(p)$ then

$$\sum_{i \in K} p \cdot \boldsymbol{a}^i < \sum_{i \in K} p \cdot \boldsymbol{x}^i.$$

It means, due to (3), that

$$\nu_{\succeq}\left(\sum_{i \in K}^{\oplus} q \cdot \boldsymbol{a}^i, \sum_{i \in K}^{\oplus} q \cdot \boldsymbol{a}^i\right) < 1$$

and, consequently, $\beta_{K,\boldsymbol{x}}(q) < 1$. □

Let us consider a set of coalitions $\mathcal{M} \subset 2^I$ such that

$$\bigcup_{K \in \mathcal{M}} K = I. \tag{35}$$

Wishing to construct an analogy of $\mathcal{M}$-equilibrium (see Subsection 2.4, formulas (16), (17)), we have to keep in mind that its generalization defined in the environment of fuzzy market is to be fuzzy. It means that, analogously to the fuzzy competitive equilibrium (23), the *fuzzy $\mathcal{M}$-equilibrium* is determined as a fuzzy subset of $\boldsymbol{X} \times P^F$ with membership function $\rho_{\mathcal{M}} : \boldsymbol{X} \times P^F$ such that for every $\boldsymbol{x} \in \boldsymbol{X}$ and $q \in P^F$

$$\rho_{\mathcal{M}}(\boldsymbol{x}, q) = \min\left[\beta_{\mathcal{M}}(\boldsymbol{x}, q), \delta_{\mathcal{M}}(\boldsymbol{x}, q)\right], \tag{36}$$

where

$$\beta_{\mathcal{M}}(\boldsymbol{x}, q) = \min\left(\beta_{K,q}(\boldsymbol{x}) : K \in \mathcal{M}\right) \tag{37}$$

denotes the possibility that $\boldsymbol{x}$ belongs to all fuzzy budget sets ${}^F\!\mathcal{B}^K(q)$, and

$$\delta_{\mathcal{M}}(\boldsymbol{x}, q) = \min\Big[\beta(\boldsymbol{y}, q), \min\Big(\nu_{\succeq}(u_K^F(\boldsymbol{x}), u_K^F(\boldsymbol{y})) : K \in \mathcal{M}\Big) : \boldsymbol{y} \in \boldsymbol{X}\Big] \tag{38}$$

denotes the possibility that

$$u_K^F(\boldsymbol{x}) \succeq u_K^F(y) \quad \text{for all } K \in \mathcal{M},\ \boldsymbol{y} \in {}^F\!\mathcal{B}^K(q).$$

Then (36) is the possibility that both above properties are fulfilled for given $\boldsymbol{x} \in \boldsymbol{X}$ and $q \in P^F$. The concept of fuzzy $\mathcal{M}$-equilibrium generalizes the traditional fuzzy competitive equilibrium. Its interpretation is similar to the interpretation of the deterministic $\mathcal{M}$-equilibrium briefly described in Subsection 2.4, with natural modifications related to the fact that the utilities and prices in the considered model are fuzzy quantities.

Remark 10. Let $\mathcal{M} = \{\{i\}\}_{i\in I}$ be the set of all one-element coalitions. Then a fuzzy state $(\boldsymbol{x}, q) \in \boldsymbol{X} \times P^F$ of $\boldsymbol{M}^F$ is a fuzzy $\mathcal{M}$-equilibrium iff it is fuzzy competitive equilibrium, as follows from (23), (24), (25) and (36), (37), (38) with regard to (22) and (34).

Remark 11. Let $\mathcal{M} \subset 2^I$ and $\mathcal{N} \subset 2^I$ fulfil

$$\bigcup\nolimits_{K\in\mathcal{M}} K = \bigcup_{k\in\mathcal{N}} K = I.$$

Let, moreover, $\mathcal{M} \subset \mathcal{N}$. Then a fuzzy state $(\boldsymbol{x}, q)$ is a fuzzy $\mathcal{M}$-equilibrium if it is fuzzy $\mathcal{N}$-equilibrium, i. e.,

$$\rho_{\mathcal{M}}(\boldsymbol{x}, q) \geq \rho_{\mathcal{N}}(\boldsymbol{x}, q).$$

Theorem 8. Let $\boldsymbol{M}^F$ be fuzzy extension of $\boldsymbol{M}$ such that $\mu_{i,\boldsymbol{x}}(r) = 1$ iff $r = u_i(\boldsymbol{x})$ and let $q \in P^F$ be fuzzy extension of $p \in P$, i. e., $\pi_j(r) = 1$ iff $r = p_j$, $j = 1, \dots, m$. Let $\mathcal{M} \subset 2^I$ fulfil (35). Then fuzzy $\mathcal{M}$-equilibrium $(\boldsymbol{x}, q)$ of $\boldsymbol{M}^F$ is fuzzy extension of the $\mathcal{M}$-equilibrium of $\boldsymbol{M}$. It means that $\rho_{\mathcal{M}}(\boldsymbol{x}, q) = 1$ if and only if $(\boldsymbol{x}, p)$ is $\mathcal{M}$-equilibrium.

P r o o f. The theorem follows from the previous two lemmas and definitions (23) and (36). If $(\boldsymbol{x}, p)$ is $\mathcal{M}$-equilibrium in $\boldsymbol{M}$ then by (37), $\beta_{\mathcal{M}}(\boldsymbol{x}, q) = 1$ and by (38) $\delta_{\mathcal{M}}(\boldsymbol{x}, q) = 1$. If $(\boldsymbol{x}, p)$ is not the $\mathcal{M}$-equilibrium then either $\boldsymbol{x} \in \mathcal{B}^K(p)$ for some $K \in \mathcal{M}$ which means that $\beta_{\mathcal{M}}(\boldsymbol{x}, q) < 1$ as follows from Lemma 5, or $u_K(\boldsymbol{x}) < u_K(\boldsymbol{x}')$ for some $\boldsymbol{x} \in \mathcal{B}_p^K$ and then (3) implies that $\delta_{\mathcal{M}}(\boldsymbol{x}, q) < 1$. □

Lemma 5. Let $\boldsymbol{M} = (I, m, \boldsymbol{a}, (u_i^F)_{i\in I})$ and $\overline{\boldsymbol{M}}^F = (I, m, \boldsymbol{a}, (\overline{u}_i^F)_{i\in I})$ be two fuzzy extensions of market $\boldsymbol{M}$ such that $\mu_{i,\boldsymbol{x}}$, $\overline{\mu}_{i,\boldsymbol{x}}$ are membership functions of $u_i^F(\boldsymbol{x})$, $\overline{u}_i^F(\boldsymbol{x})$, respectively, where $\mu_{i,\boldsymbol{x}}(r) \geq \overline{\mu}_{i,\boldsymbol{x}}(r)$ for all $r \in R$. Let q, $\overline{q} \in P^F$ be fuzzy extensions of $p \in P$ with $\pi_j(r) \geq \overline{\pi}_j(r)$ for all $r \in R$, $j = 1, \dots, m$. Let for some $\mathcal{M} \subset 2^I$, $\rho_{\mathcal{M}}$, $\overline{\rho}_{\mathcal{M}}$ be the membership functions of fuzzy $\mathcal{M}$-equilibria in markets $\boldsymbol{M}^F$ and $\overline{\boldsymbol{M}}^F$. Then for every $\boldsymbol{x} \in \boldsymbol{X}$

$$\rho_{\mathcal{M}}(\boldsymbol{x}, q) \geq \overline{\rho}(\boldsymbol{x}, \overline{q}).$$

P r o o f. The statement follows from (36), (37) and (38). The relation between π_j and $\overline{\pi}_j$ means, via (3) and (34) that $\beta_{\mathcal{M}}(\boldsymbol{x}, q) \geq \overline{\beta}_{\mathcal{M}}(\boldsymbol{x}, \overline{q})$. The monotonicity with the previous inequality, and with relation between $\mu_{K,\boldsymbol{x}}$ and $\overline{\mu}_{K,\boldsymbol{x}}$ means that $\delta_{\mathcal{M}}(\boldsymbol{x}, q) \geq \overline{\delta}_{\mathcal{M}}(\boldsymbol{x}, \overline{q})$ where $\overline{\beta}_{\mathcal{M}}$ and $\overline{\delta}_{\mathcal{M}}$ are the functions defined by (37) and (38) for $\overline{\boldsymbol{M}}^F$. These two inequalities prove the statement. □

Remark 12. Let us note that if $(\boldsymbol{x}, p)$ is an $\mathcal{M}$-equilibrium of a deterministic market $\boldsymbol{M}$, if $\boldsymbol{M}^F$ and $\overline{\boldsymbol{M}}^F$ are fuzzy extensions of $\boldsymbol{M}$ and q, $\overline{q}$ are fuzzy extensions of p then the inequality in Lemma 5 turns into equality as follows from the previous results, namely from Theorem 8.

It can be seen that some very special cases of the set of coalitions $\mathcal{M} \subset 2^I$ considered in the concept of $\mathcal{M}$-equilibrium generate some specific properties of the fuzzy $\mathcal{M}$-equilibria. For example, if $\mathcal{M}$ is a coalition structure, i. e.,

$$\mathcal{M} = \{K_1, K_2, \ldots, K_\ell\}$$

such that $K_j \cap K_k = \emptyset$ for $j \neq k$, $K_1 \cup K_2 \cup \cdots \cup K\ell = I$, then the fuzzy $\mathcal{M}$-equilibrium is formally identical with the fuzzy competitive equilibrium of a fuzzy competitive market

$$\boldsymbol{M}^F_{\mathcal{M}} = (\mathcal{M}, m, \boldsymbol{a}_{\mathcal{M}}, (u_K)_{K \in \mathcal{M}})$$

where $\mathcal{M}$ is the set of ℓ "collective" agents $K_1, \ldots, K_\ell$ and

$$\boldsymbol{a}_{\mathcal{M}} = (\boldsymbol{a}^K)_{K \in \mathcal{M}}, \quad \boldsymbol{a}^K_j = \sum\nolimits_{i \in K} a^i_j, \quad j = 1, \ldots, m.$$

Moreover, if $\mathcal{M} = \{I\}$ contains exactly one coalition then $(\boldsymbol{x}, q)$ is a fuzzy $\mathcal{M}$-equilibrium iff $\boldsymbol{x} \in \boldsymbol{X}^I$, i. e.

$$\sum\nolimits_{i \in I} x^i_j \leq \sum\nolimits_{i \in I} a^i_j, \quad j = 1, \ldots, m$$

(cf. (7)), and for all $\boldsymbol{y} \in \boldsymbol{X}^I$

$$\sum\nolimits^{\oplus}_{i \in I} u_i(\boldsymbol{x}) \geq \sum\nolimits^{\oplus}_{i \in I} u^F_i(\boldsymbol{y}).$$

This extremal case ignores all market elements of the modelled situation. The influence of the prices (no matter if they are crisp or fuzzy) is completely eliminated, the budget sets are without any sense. The behaviour of the agents is reduced to collective unlimited exchange of goods aiming to maximize the sum of the individual utilities (with eventual transfer of those utilities by means of some universal representation, e. g. money).

3.4 Fuzzy $\mathcal{M}$–Core

The concept of fuzzy market itself, including the fuzzy prices, is the one analyzed in Subsection 3.1. It means that also the fuzzy market game derived from that market keeps identical with the one considered in Subsection 3.2. Nevertheless, special stress on only some coalitions expressed by the concept of fuzzy $\mathcal{M}$-equilibrium, motivates the question if the same stress on some coalitions influences the model of fuzzy game, namely its core. The adequate modification of the concept of core can be formalized in the following way.

Let $\mathcal{M}$ be a set of coalitions fulfilling (35) and let (I, w) be fuzzy market game derived from fuzzy market $\boldsymbol{M}^F$. Then we say that a fuzzy set of imputations $C_w^{\mathcal{M}} \in \mathcal{F}(R^n)$ with membership function $\gamma_{\mathcal{M}} : R^n \to [0, 1]$ is a *fuzzy $\mathcal{M}$-core* of (I, w) iff for $\boldsymbol{r} \in R^n$

$$\gamma_{\mathcal{M}}(\boldsymbol{r}) = \min \left(\nu_{\succeq} \left(w(I), \left\langle \textstyle\sum_{i \in I} r_i \right\rangle \right), \, \widehat{\gamma}_{\mathcal{M}}(\boldsymbol{r}) \right), \tag{39}$$

where

$$\widehat{\gamma}_{\mathcal{M}}(\boldsymbol{r}) = \min \left(\nu_{\succeq} \left(\textstyle\sum_{i \in K} r_i, \, w(K) \right) : K \in \mathcal{M} \right),$$

(cf. (32)). The interpretation of the above formulas is obvious – the imputation from the $\mathcal{M}$-core is to be accessible for the coalition of all agents, and it is not to be blocked by any significant coalition (i. e., coalition from $\mathcal{M}$). In the fuzzified environment the possibilities that these demands are satisfied are given by the values of membership functions.

Lemma 6. If $\mathcal{M} = 2^I$ is the set of all coalitions $K \subset I$, then $C_w^{\mathcal{M}} = C_w$, i. e., $\gamma_{\mathcal{M}}(\boldsymbol{r}) = \gamma(\boldsymbol{r})$ for all $\boldsymbol{r} \in R^n$.

Proof. Comparing (39) and (32), it is easy to see that the value $\nu_{\succeq} (w(I), \langle \sum_{i \in I} r_i \rangle)$ is always identical for both definitions. The values $\widehat{\gamma}(\boldsymbol{r})$ and $\widehat{\gamma}_{\mathcal{M}}(\boldsymbol{r})$ are identical if $\mathcal{M} = 2^I$. $\square$

Remark 13. It is easy to see that $\mathcal{N} \subset \mathcal{M} \subset 2^I$ are sets of coalitions then

$$C_w^{\mathcal{M}} \subset C_w^{\mathcal{N}}, \quad \text{i. e.,} \quad \gamma_{\mathcal{M}}(\boldsymbol{r}) \le \gamma_{\mathcal{N}}(\boldsymbol{r}) \quad \text{for all } \boldsymbol{r} \in R^n.$$

Remark 14. Obviously, if $\mathcal{M} = \{I\}$ is the set containing exactly one coalition of all agents then

$$\gamma_{\{I\}}(\boldsymbol{r}) = \min \left(\nu_{\succeq} \left(w(I), \left\langle \textstyle\sum_{i \in I} r_i \right\rangle \right), \nu_{\succeq} \left(\left\langle \textstyle\sum_{i \in I} r_i \right\rangle, w(I) \right) \right).$$

Due to the definition (28) of $w(I)$, there in such case always exists $\boldsymbol{r} \in R^n$ such that $\gamma_{\{I\}}(\boldsymbol{r}) = 1$.

Lemma 7. Let $\mathcal{K} = \{K_1, \dots, K_k\}$ be coalitional structure (i. e., $K_j \cap K_\ell = \emptyset$ for $j \ne \ell$, $K_1 \cup \cdots \cup K_\ell = I$, let $\mathcal{M} = \{I, K_1, \dots, K_\ell\}$ and let $\mathcal{N} = \{I\}$). Then for every $\boldsymbol{r} \in R^n$ $\gamma_{\mathcal{M}}(\boldsymbol{r}) = \gamma_{\mathcal{N}}(\boldsymbol{r})$.

Proof. Due to Remark 13, $\gamma_{\mathcal{M}}(\boldsymbol{r}) \le \gamma_{\mathcal{N}}(\boldsymbol{r})$ for any $\boldsymbol{r} \in R^n$. It is evident that the value

$$\nu_{\succeq} \left(w(I), \left\langle \textstyle\sum_{i \in I} r_i \right\rangle \right)$$

is identical for $\gamma_{\mathcal{M}}$ and $\gamma_{\mathcal{N}}$. Let us turn our attention to $\widehat{\gamma}_{\mathcal{M}}$ and $\widehat{\gamma}_{\mathcal{N}}$. Due to Theorem 4, the fuzzy market game (I, w) is fuzzy superadditive. It means that (cf. (30))

$$\nu_{\succeq} \left(w(I), \, w(K_1) \oplus \cdots \oplus w(K_k) \right) = 1.$$

It means that for any $\boldsymbol{r} \in R^n$

$$\begin{aligned}
& \nu_{\succeq}\left(\left\langle \sum_{i\in I} r_i \right\rangle,\, w(I)\right) \\
\geq\ & \nu_{\succeq}\left(\left\langle \sum_{i\in K_1} r_i \right\rangle \oplus \cdots \oplus \left\langle \sum_{i\in K_l} r_i \right\rangle,\, w(K_1) \oplus \cdots \oplus w(K_k)\right) \\
\geq\ & \min\left(\left\langle \sum_{i\in K_\ell} r_i \right\rangle,\, w(K_\ell) : \ell = 1, 2, \ldots, k\right).
\end{aligned}$$

Hence, $\widehat{\gamma}_{\mathcal{N}}(\boldsymbol{r}) = \widehat{\gamma}_{\mathcal{M}}(\boldsymbol{r}) = \nu_{\succeq}\left(\left\langle \sum_{i\in I} r_i \right\rangle,\, w(I)\right)$. It means that the stated equality is valid. □

The relation between fuzzy $\mathcal{M}$-core of a market game and $\mathcal{M}$-equilibrium of a deterministic market is similar to the relation between competitive equilibrium and fuzzy core of fuzzy market game, formulated in Theorem 5.

Theorem 9. Let $\boldsymbol{M}$ be a competitive market, let $\boldsymbol{M}^F$ be its fuzzy extension, let (I, w) be fuzzy market game of $\boldsymbol{M}^F$, and let $\mathcal{M}$ be a set of coalitions fulfilling (35). If for some $\boldsymbol{r} \in R^n$, $\boldsymbol{r} \in C_v^{\mathcal{M}}$ (see (18)) then $\gamma_{\mathcal{M}}(\boldsymbol{r}) = 1$.

Proof. The proof is analogous to the proof of Theorem 4. If $\boldsymbol{r} \in C_v^{\mathcal{M}}$ then

$$\sum_{i\in I} r_i \leq v(I) \quad \text{and} \quad \sum_{i\in K} r_i \geq v(K) \quad \text{for all } K \subset \mathcal{M}.$$

As w is fuzzy extension of v (cf. Theorem 3),

$$\nu_{\succeq}\left(w(I), \left\langle \sum_{i\in I} r_i \right\rangle\right) = 1$$

and

$$\nu_{\succeq}\left(\left\langle \sum_{i\in K} r_i \right\rangle,\, w(K)\right) = 1 \quad \text{for all } K \in \mathcal{M}$$

and $\chi_K(v(K)) = 1$ for any $K \subset I$ (where χ_K is the membership function of $w(K)$). The above equalities prove the statement. □

4 Alternative Approaches and Observations

The mathematical theory of cooperation is, more or less openly, based on the theory of games and its particular approaches. Recently, those approaches are widely developed in numerous specific branches. The one, called cooperative games with transferable utility (TU-games), was used in the previous sections to model the market. Even this seemingly narrow branch of game theory has its modifications. For example, the expected outputs can be fuzzified, as shown in Section 3 and in [12, 9], e. g. But the fuzzification of outputs has its alternative construction [15] or the fuzzification may be regarded to the structure of coalitions (see [1, 2, 3, 14], e. g.) or the fuzzy coalitions may be formed by compact blocks of cooperating individuals (see [13] and partly also [14]).

In this section, we briefly discuss the possibility of alternative approaches to the fundamental market model $\boldsymbol{M}$ formulated in Subsection 2.3, which alternatives are inspired by some modifications of the fuzzy cooperative games. The aim of the following comments is not to offer a deep and thorough analysis of the alternative approaches but to turn attention to their existence, show their essential construction, eventually to point at some usually obvious observations connected with their properties.

4.1 Fuzzy Classes of Crisp Markets

This modification of the fuzzy market model is inspired by the method suggested for TU-games with vague characteristic functions in [15] and several related papers. It is to be admitted that the seemingly more natural fuzzification of the characteristic function v in the game (I, v), by fuzzy function $w : 2^I \to \mathcal{F}(R)$, where $w(K)$ are fuzzy quantities (see Subsection 2,2 and [12]), leads to some essential difficulties. They are described and analyzed in [12]. On the other hand, the fuzzification suggested in [15] is flexible and lucid enough, and it offers to use the knowledge of well developed theory of crisp TU-games. Its main principle is based on the idea not to fuzzify particular components of the game (eventually market) but to operate with fuzzy classes of completely crisp games (markets). In the models considered in this chapter, that principle can be realized as follows.

Let us denote by $\boldsymbol{U}(I, m, \boldsymbol{a})$ or briefly only $\boldsymbol{U}$ the set of all competitive markets over the set of agents I, number of goods m and initial distribution of goods $\boldsymbol{a}$, i. e.,

$$\boldsymbol{U}(I, m, \boldsymbol{a}) = \Big\{\boldsymbol{M} = (I, m, \boldsymbol{a}, (u_i)_{i\in I}) : u_i \text{ are not-decreasing and concave utility functions, fulfilling (8)}\Big\}. \tag{40}$$

Let us consider a deterministic competitive market $\overline{\boldsymbol{M}} \in \boldsymbol{U}$ with utility functions $\overline{u}_i$, $i \in I$. Let us, further, consider fuzzy extension $\overline{\boldsymbol{M}}^F$ of $\overline{\boldsymbol{M}}$ with fuzzy utilities $\overline{u}_i^F$, where fuzzy quantities $\overline{u}_i^F(\boldsymbol{x})$ have membership functions $\overline{\mu}_{i,\boldsymbol{x}} : R \to [0, 1]$, as introduced in Subsection 3.1 in (19) and (20). Let us construct a fuzzy subset of $\boldsymbol{U}$ with membership function $\psi : \boldsymbol{U} \to [0, 1]$ such that for every competitive market $\boldsymbol{M} \in \boldsymbol{U}$ the value $\psi(\boldsymbol{M})$ represents the possibility that fuzzy market $\overline{\boldsymbol{M}}^F$ achieves the values of its fuzzy utilities $\overline{u}_i^F(\boldsymbol{x})$ identical with the utilities $u_i(\boldsymbol{x})$ of market $\boldsymbol{M}$; in symbols,

$$\psi(\boldsymbol{M}) = \min\Big(\overline{\mu}_{i,\boldsymbol{x}}(u_i(\boldsymbol{x})) : i \in I,\, \boldsymbol{x} \in \boldsymbol{X}\Big). \tag{41}$$

In this way, we have transformed the fuzzy market $\overline{\boldsymbol{M}}^F$ into a fuzzy class of crisp competitive markets. Each of those deterministic markets is well managed in the literature, its budget sets, states of market and competitive equilibria are well described and characterized. This knowledge can be transformed into the fuzzy class of such markets. Let us start with simple and quite obvious observations.

Observation 1. The above construction may be used for every fuzzy market $\boldsymbol{M}^F$.

Observation 2. Let $\boldsymbol{M}^F$ be fuzzy extension of $\boldsymbol{M}$ in the sense that for all $i \in I$ and $\boldsymbol{x} \in \boldsymbol{X}$, $\mu_{i,\boldsymbol{x}}(u_i(\boldsymbol{x})) = 1$. Then $\psi(\boldsymbol{M}) = 1$.

Observation 3. Let $\overline{\boldsymbol{M}} \in \boldsymbol{U}$ and let $\overline{\boldsymbol{M}}^F$, $\widetilde{\boldsymbol{M}}^F$ be its fuzzy extensions with utilities $\overline{u}_i$, $\tilde{u}_i$ and their membership functions $\overline{\mu}_{i,\boldsymbol{x}}$, $\tilde{\mu}_{i,\boldsymbol{x}}$, $i \in I$, $\boldsymbol{x} \in \boldsymbol{X}$, respectively. Let for every $r \in R$, $\overline{\mu}_{i,\boldsymbol{x}}(r) \geq \tilde{\mu}_{i,\boldsymbol{x}}(r)$. If we denote $\overline{\psi}$ and $\tilde{\psi}$ the membership functions (41) constructed for $\overline{\boldsymbol{M}}$ and $\widetilde{\boldsymbol{M}}^F$, respectively, then

$$\overline{\psi}(\boldsymbol{M}) \geq \tilde{\psi}(\boldsymbol{M}) \quad \text{for all } \boldsymbol{M} \in \boldsymbol{U}.$$

Analogous procedure which was shown for fuzzy markets can be used for fuzzy prices. In fact, the simple structure of P and P^F makes the formal construction much more easy and lucid. It is worth remembering that certain preliminary comments of this type were already done in one of the previous sections, namely in Subsection 3.1, Theorem 2.

Let us consider vector of fuzzy prices $q = (q_j)_{i=1,\dots,m}$, where for each $j \in \{1, \dots, n\}$, $\pi_j : R_+ \to [0,1]$ is the membership function of q_j. Let us construct fuzzy subset of P with membership function $\pi : P \to [0,1]$ such that

$$\pi(p) = \min\left(\pi_j(p_j) : j = 1, \dots, m\right), \quad p \in P. \tag{42}$$

Each of the crisp markets from $\boldsymbol{U}$ has its set of its states $(\boldsymbol{x}, p)$, $p \in P$. Also the fuzzy market $\boldsymbol{M}^F$, represented by fuzzy class of crisp markets with ψ, together with the fuzzy set of (crisp) price vectors with π (cf., (41), (42)) has states of markets which form a fuzzy subset of $\boldsymbol{X} \times P$ with membership function $\xi : \boldsymbol{X} \times P \to [0,1]$, where

$$\xi(\boldsymbol{x}, p) = \max\left(\psi(\boldsymbol{M}) : \boldsymbol{M} \in \boldsymbol{U},\ (\boldsymbol{x}, p) \text{ is a state of market } \boldsymbol{M}\right). \tag{43}$$

Observation 4. It is not difficult to derive that $\xi(\boldsymbol{x}, p) = \pi(p)$ for $\boldsymbol{x} \in \boldsymbol{X}$(cf. Theorem 2).

Very similarly, we may define fuzzy competitive equilibria of fuzzy market $\boldsymbol{M}^F$ represented by fuzzy subclass of $\boldsymbol{U}$ with ψ (see (41)) as a fuzzy set of (crisp) equilibria of markets from $\boldsymbol{U}$. We denote its membership function by $\rho^* : \boldsymbol{X} \times P \to [0,1]$, where for $(\boldsymbol{x}, p)$

$$\rho^*(\boldsymbol{x}, p) = \max\left(\psi(\boldsymbol{M}) : \boldsymbol{M} \in \boldsymbol{U},\ (\boldsymbol{x}, p) \text{ is competitive equilibrium of } \boldsymbol{M}\right). \tag{44}$$

The methodological principles of this approach are quite understandable – to transfer the problem from fuzzified components of an individual market to fuzzy class of crisp (and, consequently, well managed) markets. For example, the market game of the fuzzy market $\boldsymbol{M}^F$represented by fuzzy subclass of $\boldsymbol{U}$ with ψ can be constructed as a fuzzy class of deterministic games Γ_I (see Subsection 3.2)with membership function $\chi^* : \Gamma_I \to [0,1]$, where for any TU-game (I, v), where

$$\chi^*(v) = \max\left(\psi(\boldsymbol{M}) : (i, v) \text{ is the market game of } \boldsymbol{M}\right). \tag{45}$$

The challenge of this alternative approach to fuzziness in a competitive market and market equilibrium consists in the analysis of the possibilities (and limits) of the exploation of the well known properties of the crisp markets, namely the relation between their equilibria and the cores of related market games. Their extension from results valid for individual markets and TU-games to the whole fuzzy classes of such objects looks attractive but it is not selfevident.

4.2 Fuzzy Coalitions

If the pattern for mathematical models of cooperative activities is to be found in the theory of cooperative games (and it appears to be natural)then we must stress the fact that the main stream of fuzzification of TU-games regards the fuzzification of coalitions (see, e. g., [1, 2, 3, 4, 13, 14]). The TU-game model includes two main elements – the set of players I, with the concept of coalition following from it, and the characteristic function v representing the motivation of players. The fuzzification of the characteristic function is investigated since ninetieths (see, e. g., [12, 15])and the experience with this fuzzification was several times quoted in [9] and in the previous sections of this paper. The vagueness of the structure of cooperative bounds among players is investigated in numerous works. It is based on the idea that some players may participate in several coalitions, parallelly. They distribute their "power" (whatever it means in each actual situation – time, endeavour, financial means, etc.) into several activities. Such vague structure of cooperation is formally described by fuzzy coalitions.

A *fuzzy coalition* L is usually defined as a fuzzy subset of I with membership function

$$\tau_L : I \to [0, 1]$$

where $\tau_L(i)$ is the number denoting which part of his "power"offers player i to the coalition L. It means that this value is relative (related to the complete possibilities of the player)and it may represent different absolute values. The theory of TU-games with fuzzy coalitions is relatively deeply elaborated and it is still developed (cf. [3, 4]). On the other hand, the eventual interpretation of some of its concepts may be connected with some degree of subjectivity. It follows from the fact that in certain sense, each fuzzy coalition regards all players from I, even if some of them with possibility $\tau_L(i) = 0$. But also for players with positive membership, that value may vary in the interval $(0, 1]$ and it is sometimes difficult to decide if the participation with very small value $\tau_L(i)$ is significant. Consequences of such hesitation may be reflected if some concepts which are clear for the crisp TU-games, like, e. g., disjointness of coalitions and, consequently, supeadditivity, additivity and coalitional structure, are to be extended in the environment of TU-games with fuzzy coalitions (cf., [14] and also [13]).

On the other hand, the theory of TU-games with fuzzy coalitions frequently does not deeply respect the intuitive presumption that fuzzy coalitions extend the cooperative relations determined by crisp coalition and that this fact is to be reflected by adequate relations between the values of pay-offs $v(L)$. A natural form of this relation was suggested in [14]. It follows from the fact that each fuzzy coalition L can be represented by a convex combination of crisp coalitions $K_0, K_1, \ldots, K_N \in \boldsymbol{K}$ in the sense that for each player $i \in I$, the value $\tau_L(i)$ is a convex combination of $\tau_{K_0}(i), \ldots, \tau_{K_N}(i)$ achieving only values from $\{0, 1\}$.

This obvious conclusion may be generalized, and it is possible to represent fuzzy coalition L by a fuzzy subset of the set of all crisp coalitions. This representation of fuzzy coalitions is essentially different from the original representation by fuzzy sets of players. The definitions of many fundamental concepts (like core) differ, and some properties, even if they in some specific way reflect well known concepts of the theory of games with crisp coalitions, are based on different principles. More about this topic can be found in [13]. Summarizing those differences, it is possible to conclude that each fuzzy coalition represented by fuzzy set of players may be converted into the representation by fuzzy set of crisp coalitions but not vice-versa. Moreover, some of fuzzy coalitions represented by fuzzy sets of players may be converted in several different representations by fuzzy sets of crisp subcoalitions. It means that the representation of fuzzy cooperation by fuzzy sets of crisp coalitions is more sophisticated and offers more possibilities to reflect the fine variations of the cooperation inside the vague coalitions.

From the point of view of this paper, it is important to find an adequate relation between the concept of fuzzy coalition and the modelled reality of the market exchange. The solution of this problem reaches over the main topic of this chapter and demands deep analysis and principally new approaches to the problems including re-formulation of the model. It is useful to mention the basic facts:

— The free exchange in the classical market model (Subsection 2.3) is formulated as a purely individual activity. Each agent has its own vector of goods $\boldsymbol{x}^i$, the budget sets $\mathcal{B}^i(p)$ are defined and processed exclusively as characteristics of individual agents, and also the equilibria are defined from the individual point of view.

— On the other hand – the process of the exchange is in it essence a cooperative matter – each agent needs some partners with which he exchanges his initial vector of goods $\boldsymbol{a}^i$. These partners may be some (not necessarily all) members of I and the exchange with different of them (or their groups)may be realized on different levels.

— It is natural and usual that an agent distributes his "power"(i. e., the disposable goods) among several partners to achieve the maximal utility of the result of exchange.

— The relation between equilibrium of a market and core of cooperative market game, which belongs to the most important results of the referred market theory (see, e. g. [5, 6, 16, 17, 18] and also [7, 8]) shows that there exists an essential, even if not spectacular, relation between the processes forming the competitive market and cooperative game.

If the methodology of fuzzy coalitions is to be included into the market model, it will be necessary to take the above facts into account and to respect them in the modified model. They show the fact evident from the very roots of the market model – the market exchange from its very beginning displays some essential features of the distribution of agents' power in (fuzzy?) coalitions.

Very preliminarily and with a large probability that some serious problems will be met, the modification of the classical market model which was suggested in Subsection 2.4 and which introduces the concept of $\mathcal{M}$-equilibria, seems to offer one of the potentially possible ways how to include fuzzy coalitions in the market model. It is possible to extend the set of coalitions $\mathcal{M}$ by (at least some) fuzzy coalitions and to verify the consequences of that modification for the structure of coalitional budget sets and corresponding $\mathcal{M}$-equilibria.

More reliable conclusions in this topic may follow only from eventual more detailed analysis of such model in the future.

5 Conclusions

In the previous sections, we have briefly recollected some basic concepts of the competitive market and suggested its fuzzification consisting in the fuzzification of utilities and prices.

We have also discussed, at least at a heuristic level, the possibility of the fuzzification of coalitions in such market as a qualitatively new approach to the investigation of market.

It is necessary to stress the fact that the basic market model considered here is the most simplified one, in order not to complicate the procedure of its fuzzification (and the necessary formalism which is quite complicated by itself). Namely, we have completely omitted the possibility to exclude one very specific type of goods, called *money*, which may exist in negative quantities and which serves as a universal (and for all agents equal) linear representative of utility (see [5, 7, 8, 16, 17] and many others). The extension of the presented model by "money"is interesting, inavoidable in the future, but not essential for the first attempt to fuzzify the market model by means of fuzzification of those its components which were fuzzified here.

List od Important Symbols

a) Fuzzy Martket Concepts, Fuzzy Quantities (fq), their Membership Functions (mf)

Symbol	Interpretation	Section (Formula)
a, b, (μ_a, μ_b)	Fuzzy quantity (in general)	2.1
$\succeq$, $(\nu_{\succeq}(\cdot,\cdot))$	Fuzzy ordering relation	2.1, (3)
$u_i^F \boldsymbol{x}$, $(\mu_{i,\boldsymbol{x}}(\cdot))$	Fuzzy utility	3.1, (19)
$q = (q_j)_{j=1,\dots,m}$, $(\pi_j(\cdot))$	Fuzzy prices	3.1
P^F	Set of fuzzy prices	3.1
$\boldsymbol{M}^F$, $\overline{\boldsymbol{M}}^F$	Fuzzy competitive market	3.1
${}^F\mathcal{B}^i(q)$, $(\beta_{i,q}(\boldsymbol{x}))$	Fuzzy budget set	3.1, (22)
$\rho(\boldsymbol{x}, q)$	Fuzzy equilibrium – mf	3.1, (23)
$\beta(\boldsymbol{x}, q)$, $\delta(\boldsymbol{x}, q)$	Components of fuzzy equilibrium	3.1, (24), (25)
$u_K^F(\boldsymbol{x})$, $(\mu_{K,\boldsymbol{x}}(\cdot))$	Fuzzy coalitional utility	3.2, (26)
$w(K)$, $(\chi_K(\cdot))$	Fuzzy characteristic function	3.2, (28)
$\sigma(w)$	Fuzzy superadditivity – mf	???
C_w, $(\gamma(\boldsymbol{r}))$	Fuzzy core	3.2, (32)
$\langle r \rangle$	Fq. reduced into one value	3.2, (31)
${}^F\mathcal{B}^K(q)$, $(\beta_{K,q}(\boldsymbol{x}))$	Coalitional fuzzy budget set	3.3, (34)
$\rho_{\mathcal{M}}(\boldsymbol{x}, q)$	Fuzzy $\mathcal{M}$-equilibrium – mf	3.3, (36)
$\beta_{\mathcal{M}}(\boldsymbol{x}, q)$, $\delta_{\mathcal{M}}(\boldsymbol{x}, q)$	Components of fuzzy $\mathcal{M}$-equilibrium	3.3, (37), (38)
$\boldsymbol{M}_{\mathcal{M}}^F$	Aggregated fuzzy market	3.3
$C_w^{\mathcal{M}}$, $(\gamma_{\mathcal{M}}(\cdot))$	Fuzzy $\mathcal{M}$-core	3.4, (39)
$\psi(\boldsymbol{M})$	Fuzzy set of markets – mf	4.1, (41)
$\xi(\boldsymbol{x}, p)$	Fuzzy set of states of market –mf	4.1, (43)
$\rho^*(\boldsymbol{x}, p)$	Fuzzy set of equilibria – mf	4.1, (44)
$\chi^*(\boldsymbol{x}, p)$	Fuzzy set of market games	4.1, (45)
L, $(\tau_L(i))$	Fuzzy coalition	4.2

b) Other Symbols

Symbol	Interpretation	Section (Formula)
R, R_+	Sets of real numbers	2, intr.
R^m, R^m_+	Sets of real vectors	2, intr.
$\mathcal{F}(\cdot)$	Set of fuzzy subsets	2, intr.
r, r_i, $\boldsymbol{r}$	Real numbers and vectors	2.1
$\oplus$	Sum of fq.	2.1, (1)
I, $i\,\{1,\dots,n\}$	Sets of agents (players), agent	2.2
v	Coalitions	2.2
K, K', L	Coalitions	2.2
$\boldsymbol{K}$	Set of all coalitions	2.2, 4.1
(I,v)	TU-game	2.2
C, C_v, $C_v^{\mathcal{M}}$	Core	2.2, 2.4, (5), (18)
$\boldsymbol{x}$	Distribution matrix	2.3
$\boldsymbol{x}^i$	Structure of property	2.3
$\{1,\dots,m\}$	Set of goods	2.3
$\boldsymbol{a}$	Initial distribution matrix	2.3, (6), (7)
$\boldsymbol{X}$, $\boldsymbol{X}^K$	Sets of distribution matrices	2.3, (6), (7)
u_i, u_K	Utility functions (crisp)	2.3, (8), 2.4, (14)
$\boldsymbol{M}$, $\overline{\boldsymbol{M}}$	Market crisp	2.3, (9)
$p=(p_j)_{j=1,\dots,m}$	Vector of prices, prices (crisp)	2.3
$(\boldsymbol{x},p)$	State of market	2.3
$\mathcal{B}^i(p)$, $\mathcal{M}^K(p)$	Budget sets (crisp)	2.3, (10), 2.4, (15)
$\mathcal{M}$, $\mathcal{N}$	Sets of coalitions	2.4
$\mathcal{K}$	Coalitional structure	3.2
(I,w)	Fuzzy TU-game	3.2
Γ_I	Set of all fuzzy TU-games over I	3.2
$\sum^{\oplus}$	Summation of fq.	3.3, (33)
$\boldsymbol{U}$, $\boldsymbol{U}(I,m,\boldsymbol{a})$	Set of markets	4.1, (40)

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