Distance-based pruning for Gaussian sum method in non-Gaussian system state estimation

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Outline

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Motivation

- Gaussian Sum (GS) filter is a very popular tool in nonlinear state estimation.
- Represents a bank of Kalman filters (for linear nongaussian system) or a bank of extended Kalman filters (for nonlinear nongaussian system)
- Considering nongaussian noises, the number of Kalman filters usually grows *exponentialy* $\Rightarrow$ application of a reduction technique is *inevitable*.
- The reduction technique can significantly affect estimate quality.

Present approaches to reduction in GS

Pruning: cutting off some terms of the GS
e.g. maximum posterior probability pruning, threshold-based pruning

Merging: joining two or more GS terms into one
e.g. generalized pseudo Bayes approach

Optimization: finding parameters of a new GS using
a parameter optimization technique
e.g. statistical parameter optimization (EM)

Combined: combining the previous three approaches
parameter optimization and structural adaptation presented in papers by prof. Hanenbeck et al.

Goal of the paper

- To propose a *pruning technique* that analyzes importance of each GS term in a clever way utilizing more information than just the weights of the GS terms (e.g. using a distance of two pdf's).
- To design a *thrifty implementation* of the proposed technique to keep computational requirements low with respect to other simple pruning techniques.

State estimation

Consider a linear nongaussian system:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}_k \cdot \mathbf{x}_k + \mathbf{w}_k, \quad k = 0, 1, 2, \ldots \\ \mathbf{z}_k &= \mathbf{H}_k \cdot \mathbf{x}_k + \mathbf{v}_k, \quad k = 0, 1, 2, \ldots \end{aligned}$$

- $\mathbf{x}_k$ - state, $\mathbf{z}_k$ - measurement
- $\mathbf{w}_k$ and $\mathbf{v}_k$ are mutually independent white noises with $p(\mathbf{w}_k) = \sum_{n=1}^{q_k} \beta_{kn} \mathcal{N}\{\mathbf{w}_k : \hat{\mathbf{w}}_{kn}, \mathbf{Q}_{kn}\}$ $p(\mathbf{v}_k) = \sum_{m=1}^{r_k} \gamma_{km} \mathcal{N}\{\mathbf{v}_k : \hat{\mathbf{v}}_{km}, \mathbf{R}_{km}\}$ and independent of $\mathbf{x}_0$.

The aim of state estimation here is to find

the filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$, $\mathbf{z}^k = [\mathbf{z}_0^{\mathrm{T}}, \ldots \mathbf{z}_k^{\mathrm{T}}]^{\mathrm{T}}$

Gaussian Sum (GS) filter

- GS filter solves the Bayesian Recursive Relations (BRR) using a bank of Kalman filters.
- At each time instant k the filtering pdf is given as $p(\mathbf{x}_k|\mathbf{z}^k) = \sum_{j=1}^{\xi_k} \alpha_{kj} \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{kj}, \mathbf{P}_{kj})$.
- The means $\hat{\mathbf{x}}_{kj}$ and the covariance matrices $\mathbf{P}_{kj}$ are calculated using individual Kalman filters.
- The number of terms in the sum grows exponentially as $\xi_k = q_k r_k \xi_{k-1}$.
- Thus, a reduction technique has to be implemented to keep computational demands of the algorithm within reasonable bounds.

Distance-based pruning - an idea

1. to start with the approximate filtering pdf containing one term of the GS of the filtering pdf
2. to *append* other terms of the filtering pdf to the approximate filtering pdf successively
3. to *assess* significancy of the appended terms with respect to approximation quality
4. if the term is insignificant, it is discarded

Distance-based pruning - an idea

- Significancy of an appended term is measured using the Lissack Fu (LF) distance between the approximate pdf with the considered term and the original unpruned pdf

The LF distance between $p_1(\mathbf{x}_k|\mathbf{z}^k)$ and $p_2(\mathbf{x}_k|\mathbf{z}^k)$ is given as

$$L_\gamma(p_1,p_2) \triangleq \int |p_1(\mathbf{x}_k|\mathbf{z}^k) - p_2(\mathbf{x}_k|\mathbf{z}^k)|^\gamma \mathrm{d}\mathbf{x}_k,$$

where γ is chosen as $\gamma = 2$ in the paper.

- Considering $p_1(\mathbf{x}_k|\mathbf{z}^k) = \sum_{i=1}^{\xi^1} \alpha_{ki}^1 \mathcal{N}\{\mathbf{x}_k; \hat{\mathbf{x}}_{ki}^1, \mathbf{P}_{ki}^1\}$ and $p_2(\mathbf{x}_k|\mathbf{z}^k) = \sum_{j=1}^{\xi^2} \alpha_{kj}^2 \mathcal{N}\{\mathbf{x}_k; \hat{\mathbf{x}}_{kj}^2, \mathbf{P}_{kj}^2\}$,

$$L_2(p_1,p_2) = A_{11} - 2A_{12} + A_{22},$$

Distance-based pruning - an idea

where

$$A_{11} = \sum_{i=1}^{\xi^1} \sum_{j=1}^{\xi^1} \alpha_{ki}^1 \alpha_{kj}^1 \mathcal{N}\{\hat{\mathbf{x}}_{ki}^1 : \hat{\mathbf{x}}_{kj}^1, \mathbf{P}_{ki}^1 + \mathbf{P}_{kj}^1\},$$

$$A_{12} = \sum_{i=1}^{\xi^1} \sum_{j=1}^{\xi^2} \alpha_{ki}^1 \alpha_{kj}^2 \mathcal{N}\{\hat{\mathbf{x}}_{ki}^1 : \hat{\mathbf{x}}_{kj}^2, \mathbf{P}_{ki}^1 + \mathbf{P}_{kj}^2\},$$

$$A_{22} = \sum_{i=1}^{\xi^2} \sum_{j=1}^{\xi^2} \alpha_{ki}^2 \alpha_{kj}^2 \mathcal{N}\{\hat{\mathbf{x}}_{ki}^2 : \hat{\mathbf{x}}_{kj}^2, \mathbf{P}_{ki}^2 + \mathbf{P}_{kj}^2\}.$$

Distance-based pruning - algorithm

0. Initialization Sort the terms of the GS. The initial $\hat{p}(\mathbf{x}_k|\mathbf{z}^k)$ is given by the highest-weighted term. Specify r and $L_{2,k}^{\max} = L_2\left(p(\mathbf{x}_k|\mathbf{z}^k), \hat{p}(\mathbf{x}_k|\mathbf{z}^k)\right)$.
1. Term appending Append the next term to the approximate filtering pdf $\hat{p}(\mathbf{x}_k|\mathbf{z}^k)$.
2. Distance computing Compute $L_2\left(p(\mathbf{x}_k|\mathbf{z}^k), \hat{p}(\mathbf{x}_k|\mathbf{z}^k)\right)$.
3. Assessment If $L_2\left(p(\mathbf{x}_k|\mathbf{z}^k), \hat{p}(\mathbf{x}_k|\mathbf{z}^k)\right) \leq L_{2,k}^{\max} \cdot \exp(-r\xi_k^{\text{red}})$, the term is accepted Otherwise the term is rejected and discarded.

Distance based pruning - thrifty implamentation

Idea

computing as many terms as possible in advance and multiple utilization of the values for evaluation of the distance

- Construct a $\xi_k \times \xi_k$ matrix $\mathbf{L}_k$ with the elements $\mathbf{L}_k(i,j) = \alpha_{ki}\alpha_{kj}\mathcal{N}\{\hat{\mathbf{x}}_{ki} : \hat{\mathbf{x}}_{kj}, \mathbf{P}_{ki} + \mathbf{P}kj\}$. The matrix is symmetric.
- A_{11} is given by the normalized sum of all elements $\mathbf{L}_k(l_i, l_j)$ $i = 1, 2, \ldots, \xi_k^{\text{red}} j = 1, 2, \ldots, \xi_k^{\text{red}}$, where l_i is the index of the ith term of $\hat{p}(\mathbf{x}_k|\mathbf{z}^k)$ in $p(\mathbf{x}_k|\mathbf{z}^k)$.
- A_{12} is given by the normalized sum of all elements $\mathbf{L}_k(l_i, j)$ $i = 1, 2, \ldots, \xi_k^{\text{red}}, j = 1, 2, \ldots, \xi_k$.
- A_{22} is computed as the sum of all elements of $\mathbf{L}_k$.

Distance based pruning - thrifty implemantation

Example for computing A_{11}, A_{12}, A_{22}

Original unpruned pdf $p(\mathbf{x}_{|}\mathbf{z}^k)$ contains $\xi_k = 4$ sorted terms.
The pruned pdf $\hat{p}(\mathbf{x}_{|}\mathbf{z}^k)$ considers the terms number 1 and 3.

$$A_{11}: \quad \mathbf{L}_k = \begin{matrix} \mathbf{L}_k(1,1) & \mathbf{L}_k(1,2) & \mathbf{L}_k(1,3) & \mathbf{L}_k(1,4) \\ \mathbf{L}_k(2,1) & \mathbf{L}_k(2,2) & \mathbf{L}_k(2,3) & \mathbf{L}_k(2,4) \\ \mathbf{L}_k(3,1) & \mathbf{L}_k(3,2) & \mathbf{L}_k(3,3) & \mathbf{L}_k(3,4) \\ \mathbf{L}_k(4,1) & \mathbf{L}_k(4,2) & \mathbf{L}_k(4,3) & \mathbf{L}_k(4,4) \end{matrix}$$

Distance based pruning - thrifty implemantation

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Consider a simple system

$$x_{k+1} = 0.9 \cdot x_k + w_{k+1},\ z_k = x_k + v_k$$

$$p(w_k) = 0.2\mathcal{N}\{w_k : -1, 0.1\} + 0.6\mathcal{N}\{w_k : 0, 0.1\} + 0.2\mathcal{N}\{w_k : 1, 0.1\},$$

$$p(v_k) = 0.8\mathcal{N}\{v_k : 0, 0.01\} + 0.2\mathcal{N}\{v_k : 0, 10\},\ p(x_0) = \mathcal{N}\{x_0 : 0, 0.1\}$$

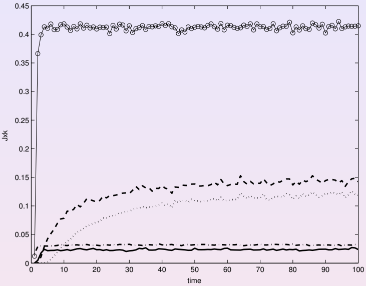
Criteria

$$J(x_k) = \mathrm{E}_{p(z^k)}\left(1 - \int \min\left[\hat{p}(x_k|z^k), p(x_k|z^k)\right] \mathrm{d}x_k\right)$$

MSE: $\bar{\Pi} = \frac{1}{100}\sum_{k=1}^{100}(x_k - \hat{x}_k)^2$

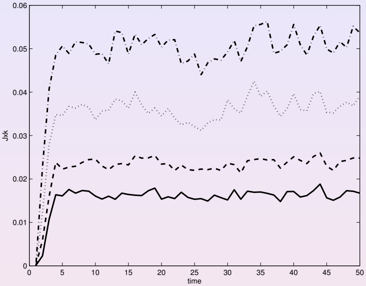
	DBP-0.4	MPPP-10	MPPP-200	EM-4	GPB1
$\bar{J}$	0.02	0.12	0.09	0.03	0.41
$\bar{\Pi}$	0.26	0.38	0.30	0.25	0.26
$\bar{T}$ [s]	0.26	0.03	0.57	11.49	0.01
$\bar{\xi}$	13.2	10	200	4	1

$J(\mathbf{x}_k)$ - various methods



DBP-0.4 - solid
MPPP-10 - dashed
MPPP-10 - dotted
EM-4 - dash-dotted
GPB-1 - circles

$J(\mathbf{x}_k)$ DBP with variable r



DBP-0.3 - dash-dotted
DBP-0.4 - dotted
DBP-0.5 - dashed
DBP-0.6 - solid

Conclusion

- A new distance-based pruning technique has been proposed.
- It prunes a term according to its significance given by the LF distance between the unpruned pdf and the pruned pdf.
- The technique provides high quality approximations with respect to the other techniques (MPPP, GPB, EM).
- The technique is very efficient due to the proposed thrifty implementation which accomplishes as many computations as possible in advance. The results of the computations are used several times for calculation of the distance.