Combining Implicational Quantifiers for Equivalence Ones by Fuzzy Connectives

Jiří Ivánek

*Institute of Information Theory and Automation**

Academy of Sciences of the Czech Republic

Pod vodárenskou věží 4, 182 08 Prague

e-mail: ivanek@utia.cas.cz

April 17, 2005

Keywords: fuzzy logic, data mining, four-fold table quantifiers.

Abstract

Relations between two Boolean attributes derived from data can be quantified by truth functions defined on four-fold tables corresponding to pairs of the attributes. Several classes of such quantifiers (implicational, double implicational, equivalence ones) with truth values in the unit interval were investigated in the frame of the theory of data mining methods. The definition of double implicational quantifiers is based on the idea of conjunction of both directions implications (similarly for equivalence). In the fuzzy logic theory, there are well-defined classes of fuzzy operators, namely t-norms representing various types of evaluations of fuzzy conjunction (and t-conorms representing fuzzy disjunction).

In the paper, it is presented that each t-norm applied to an implicational quantifier gives a double implicational quantifier. Analogously, each t-conorm applied to a double implicational quantifier gives an equivalence quantifier. Logical properties of obtained quantifiers are discussed. The method is illustrated by examples of well-known quantifiers and operators.

1 Introduction

The theory of observational quantifiers was established in the frame of the GUHA method of mechanized hypothesis formation [4]. It should be stressed that this method is one of the earliest methods of data mining. The method was during years developed and various procedures were implemented e.g. in the systems PC-GUHA [6], Knowledge Explorer [3], and 4FT-Miner [15]. Further investigations of its mathematical and logical foundations can be found in [8],

*This research has been supported by the grant 1M6798555601 of the Ministry of Education, Youth and Sports of the Czech Republic.

	ψ	$\neg\psi$
φ	a	b
$\neg\varphi$	c	d

Table 1: Four-fold table of φ and ψ

[13], [14]. We concentrate to the most widely used observational quantifiers, called in [14] four-fold table quantifiers. So far this quantifiers were treated in classical logic as 0/1-truth functions. Some possibilities of fuzzy logic approach are discussed in [8].

Several classes of quantifiers (implicational, double implicational, equivalence ones) with truth values in the unit interval are investigated in [9]. Such type of quantifications of rules derived from databases is used in modern methods of knowledge discovery in databases (see e.g. [16]). On the other hand, there is a connection between four-fold table quantifiers and measures of resemblance or similarity applied on Boolean vectors [2].

In [9], a special method of construction of double implicational quantifiers from implicational ones (and vice versa) is described. This method provides a logically strong one-to-one correspondence between classes of implicational and so called Σ-double implicational quantifiers. An analogical construction is used to introduce similar correspondence between classes of Σ-double implicational and Σ-equivalence quantifiers.

In the given paper, another method of construction is investigated. It is based on well-defined classes of fuzzy operators, namely t-norms representing various types of evaluations of fuzzy conjunction (and t-conorms representing fuzzy disjunction).

In Section 2, basic notions and classes of quantifiers are recalled, and some examples of quantifiers of different types are given. Notions of t-norms and t-conorms are defined in the frame of fuzzy logic in Section 3.

In Section 4, it is proved that each t-norm applied to an implicational quantifier gives a double implicational quantifier. The method is illustrated by examples of well-known quantifiers and operators.

Analogously, each t-conorm applied to a double implicational quantifier gives an equivalence quantifier (Section 5).

2 Classes of quantifiers

For two Boolean attributes φ and ψ (derived from given data), corresponding four-fold table $\langle a, b, c, d\rangle$ (Table 1) is composed from numbers of objects in data satisfying four different Boolean combinations of attributes:

a is the number of objects satisfying both φ and ψ,
b is the number of objects satisfying φ and not satisfying ψ,
c is the number of objects not satisfying φ and satisfying ψ,
d is the number of objects not satisfying φ and not satisfying ψ.

To avoid degenerated situations, we shall assume, that all marginals of the four-fold table are non-zero:
$a + b > 0$, $c + d > 0$, $a + c > 0$, $b + d > 0$.

Definition 1 *A four-fold table* **quantifier** $\sim$ *is an arbitrary* $[0, 1]$*-valued function defined for all four-fold tables* $\langle a, b, c, d\rangle$*. We shall write* $\sim (a, b)$ *if the value of the quantifier* $\sim$ *depends only on* a, b*;* $\sim (a, b, c)$ *if the value of the quantifier* $\sim$ *depends only on* a, b, c*;* $\sim (a, b, c, d)$ *if the value of the quantifier* $\sim$ *depends on all* a, b, c, d*.*

The most common examples of quantifiers are following ones:

Example 1 *Quantifier* $\Rightarrow_{\oslash}$ *of basic implication (corresponds to the notion of a confidence of an association rule, see [1],[4],[5]):*
$\Rightarrow_{\oslash} (a, b) = \frac{a}{a+b}$.

Example 2 *Quantifier* $\Leftrightarrow_{\oslash}$ *of basic double implication (Jaccard 1900, [2],[5]):*
$\Leftrightarrow_{\oslash} (a, b, c) = \frac{a}{a+b+c}$.

Example 3 *Quantifier* $\equiv_{\oslash}$ *of basic equivalence (Kendall, Sokal-Michener 1958, [2],[5]):*
$\equiv_{\oslash} (a, b, c, d) = \frac{a+d}{a+b+c+d}$.

If the four-fold table $\langle a, b, c, d\rangle$ represents the behaviour of the derived attributes φ and ψ in given data, then we can interpret above quantifiers in the following way:

The quantifier of basic implication calculates the relative frequency of objects satisfying ψ out from all objects satisfying φ, so it is measuring in a simple way the validity of implication $\varphi \Rightarrow \psi$ in data. The higher is a and the smaller is b, the better is validity $\Rightarrow_{\oslash} (a, b) = \frac{a}{a+b}$.

The quantifier of basic double implication calculates the relative frequency of objects satisfying $\varphi \wedge \psi$ out from all objects satisfying $\varphi \vee \psi$, so it is measuring in a simple way the validity of bi-implication $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ in data. The higher is a and the smaller are b, c, the better is validity $\Leftrightarrow_{\oslash} (a, b, c) = \frac{a}{a+b+c}$.

The quantifier of basic equivalence calculates the relative frequency of objects supporting correlation of φ and ψ out from all objects, so it is measuring in a simple way the validity of equivalency $\varphi \equiv \psi$ in data. The higher is a, d and the smaller are b, c, the better is validity $\equiv_{\oslash} (a, b, c, d) = \frac{a+d}{a+b+c+d}$.

Properties of basic quantifiers are in the core of the general definition of several useful classes of quantifiers [4], [5], [14]:
(1) I - class of implicational quantifiers,
(2) DI - class of double implicational quantifiers,
(3) ΣDI - class of Σ-double implicational quantifiers,
(4) E - class of equivalence quantifiers,
(5) ΣE - class of Σ-equivalence quantifiers.

Remark. "Equivalence quantifiers" is a term introduced by Rauch (see [14]) instead the original term "associational quantifiers" from the book by Hájek - Havránek [4].

Each class of quantifiers $\sim$ is characterized in the following definition by a special truth preservation condition of the form: fact that the four-fold table $\langle a', b', c', d' \rangle$ is in some sense (implicational, ...) better than $\langle a, b, c, d \rangle$ implies that $\sim (a', b', c', d') \geq \sim (a, b, c, d)$.

Definition 2 *Let $a, b, c, d, a', b', c', d'$ mean frequencies from arbitrary pairs of four-fold tables $\langle a, b, c, d \rangle$ and $\langle a', b', c', d' \rangle$.*

(1) A quantifier $\sim (a, b)$ is **implicational**, *$\sim \in I$, if always*
$a' \geq a \ \wedge \ b' \leq b$ *implies* $\sim (a', b') \geq \sim (a, b)$.

(2) A quantifier $\sim (a, b, c)$ is **double implicational**, *$\sim \in DI$, if always*
$a' \geq a \ \wedge \ b' \leq b \ \wedge \ c' \leq c$ *implies* $\sim (a', b', c') \geq \sim (a, b, c)$.

(3) A quantifier $\sim (a, b, c)$ is **Σ-double implicational**, *$\sim \in \Sigma DI$, if always*
$a' \geq a \ \wedge \ b' + c' \leq b + c$ *implies* $\sim (a', b', c') \geq \sim (a, b, c)$.

(4) A quantifier $\sim (a, b, c, d)$ is **equivalence**, *$\sim \in E$, if always*
$a' \geq a \wedge b' \leq b \wedge c' \leq c \wedge d' \geq d$ *implies* $\sim (a', b', c', d') \geq \sim (a, b, c, d)$.

(5) A quantifier $\sim (a, b, c, d)$ is **Σ-equivalence**, *$\sim \in \Sigma E$, if always*
$a' + d' \geq a + d \ \wedge \ b' + c' \leq b + c$ *implies* $\sim (a', b', c', d') \geq \sim (a, b, c, d)$.

Example 4 $\Rightarrow_{\oslash} \in I$, $\Leftrightarrow_{\oslash} \in \Sigma DI$, $\equiv_{\oslash} \in \Sigma E$.

Proposition 1 $I \subset DI \subset E$, $\Sigma DI \subset DI$, $\Sigma E \subset E$.

In [9], one-to-one correspondence with strong logical properties was shown
i) between classes of quantifiers $I, \Sigma DI$ by means of the relation:
$\Leftrightarrow^* (a, b, c) = \Rightarrow^* (a, b + c)$,
and, analogously,
ii) between classes of quantifiers $\Sigma DI, \Sigma E$ by means of the relation:
$\equiv^* (a, b, c, d) = \Leftrightarrow^* (a + d, b, c)$.

In the given paper, another method of construction based on fuzzy logic approach is investigated.

3 Fuzzy connectives

In fuzzy logic (for the deep theory see [7]), truth values of formulae are numbers from the unit interval (or in the general case elements of some lattice of possible values). Truth functions of connectives like conjunction and disjunction can vary from obvious min/max operators to such ones connected with some special theory or application. We shall recall the commonly used definitions of classes of operators and give several examples which will be used further.

Definition 3 *A binary operator $\otimes$ on the unit interval is called* **t-norm** *if*

(1) $\otimes$ is commutative and associative,

(2) for all x, x', y, y' such that $x' \geq x, y' \geq y$ also $x' \otimes y' \geq x \otimes y$,
(3) for all z: $z \otimes 1 = 1 \otimes z = z$.

Example 5 *Following t-norms are often used:*
Minimum t-norm:
$x \otimes_M y = min(x, y)$.
Product t-norm:
$x \otimes_P y = x \cdot y$.
Lukasiewicz's conjunction:
$x \otimes_L y = max(0, x + y - 1)$.
Pseudo-bayesian (Hamacher) t-norm (used in PROSPECTOR-like expert systems):
$x \otimes_B y = \frac{x \cdot y}{x + y - x \cdot y}$.

Definition 4 *A binary operator $\oplus$ on the unit interval is called* **t-conorm** *if*
(1) $\oplus$ is commutative and associative,
(2) for all x, x', y, y' such that $x' \geq x, y' \geq y$ also $x' \oplus y' \geq x \oplus y$,
(3) for all z: $z \oplus 0 = 0 \oplus z = z$.

Example 6 *Following t-conorms are often used:*
Maximum t-conorm:
$x \oplus_M y = max(x, y)$.
Co-product t-conorm:
$x \otimes_P y = x + y - x \cdot y$.
Lukasiewicz's disjunction:
$x \oplus_L y = min(1, x + y)$.
Pseudo-bayesian t-conorm (used in PROSPECTOR-like expert systems):
$x \oplus_B y = \frac{x \cdot (1-y) + (1-x) \cdot y}{1 - x \cdot y}$.

4 Combining implicational quantifiers for double implicational ones

Let $\Rightarrow^*$ be an implicational quantifier. There is a natural task to construct double implicational quantifiers $\Leftrightarrow^*$ using various types of fuzzy conjunction of both implications $\varphi \Rightarrow^* \psi$, $\psi \Rightarrow^* \varphi$. The proof of the following theorem is obvious; examples of its applications are, maybe, more interesting.

Theorem 1 *Let $\Rightarrow^*$ be an implicational quantifier and $\otimes$ be a t-norm. Then the quantifier $\Leftrightarrow^*$ constructed for all four-fold tables $\langle a, b, c, d \rangle$ from $\Rightarrow^*$ by the formula*
$\Leftrightarrow^* (a, b, c) = \Rightarrow^* (a, b) \otimes \Rightarrow^* (a, c)$
is a double implicational quantifier satisfying the property
$\Leftrightarrow^* (a, b, c) \leq min(\Rightarrow^* (a, b), \Rightarrow^* (a, c))$
for all four-fold tables $\langle a, b, c, d \rangle$.

Remark. From the (fuzzy) logic point of view, it means that in all models (data) the formulae $\varphi \Rightarrow^* \psi$, $\psi \Rightarrow^* \varphi$ are at least so true as the formula $\varphi \Leftrightarrow^* \psi$, i.e. deduction rules $\frac{\varphi \Leftrightarrow^* \psi}{\varphi \Rightarrow^* \psi}$, $\frac{\varphi \Leftrightarrow^* \psi}{\psi \Rightarrow^* \varphi}$ are correct.

Example 7 *For the basic implication* $\Rightarrow_{\oslash} (a,b) = \frac{a}{a+b}$*,*
the basic double implication
$\Leftrightarrow_{\oslash} (a,b,c) = \frac{a}{a+b+c}$
is the result of applying the pseudo-bayesian operator
$x \otimes_B y = \frac{x \cdot y}{x+y-x \cdot y}$*:*
$\Rightarrow_{\oslash} (a,b) \otimes_B \Rightarrow_{\oslash} (a,c) = \frac{a}{a+b} \otimes_B \frac{a}{a+c} = \frac{\frac{a}{a+b} \cdot \frac{a}{a+c}}{\frac{a}{a+b} + \frac{a}{a+c} - \frac{a}{a+b} \cdot \frac{a}{a+c}} = \frac{a}{a+b+c}$.

Example 8 *For the basic implication* $\Rightarrow_{\oslash} (a,b) = \frac{a}{a+b}$*,*
the double implication quantifier
$\Leftrightarrow_P (a,b,c) = \frac{a^2}{(a+b)\cdot(a+c)}$
is the result of applying the product operator
$x \otimes_P y = x \cdot y$*:*
$\Rightarrow_{\oslash} (a,b) \otimes_P \Rightarrow_{\oslash} (a,c) = \frac{a}{a+b} \otimes_P \frac{a}{a+c} = \frac{a}{a+b} \cdot \frac{a}{a+c} = \frac{a^2}{(a+b)\cdot(a+c)}$.

Example 9 *For the basic implication* $\Rightarrow_{\oslash} (a,b) = \frac{a}{a+b}$*,*
the double implication quantifier
$\Leftrightarrow_L (a,b,c) = max(0, \frac{a^2-bc}{(a+b)\cdot(a+c)})$
is the result of applying Lukasiewicz's operator
$x \otimes_L y = max(0, x+y-1)$
$\Rightarrow_{\oslash} (a,b) \otimes_L \Rightarrow_{\oslash} (a,c) = \frac{a}{a+b} \otimes_L \frac{a}{a+c} = max(0, \frac{a}{a+b} + \frac{a}{a+c} - 1) = max(0, \frac{a^2-bc}{(a+b)\cdot(a+c)})$.

5 Combining double implicational quantifiers for equivalency ones

Let $\Leftrightarrow^*$ be a double implicational quantifier. Similarly as in the previous section, there is a natural task to construct equivalency quantifiers $\equiv^*$ using various types of fuzzy disjunction of both double implications $\varphi \Leftrightarrow^* \psi, \neg\varphi \Leftrightarrow^* \neg\psi$. The proof of the following theorem is also obvious; examples of its applications are more complicated, so only the most interesting one is provided.

Theorem 2 *Let* $\Leftrightarrow^*$ *be a double implicational quantifier and* $\oplus$ *be a t-conorm. Then the quantifier* $\equiv^*$ *constructed for all four-fold tables* $\langle a,b,c,d \rangle$ *from* $\Leftrightarrow^*$ *by the formula*
$\equiv^* (a,b,c,d) = \Leftrightarrow^* (a,b,c) \oplus \Leftrightarrow^* (d,b,c)$
is a double implicational quantifier satisfying the property
$\equiv^* (a,b,c,d) \geq max(\Leftrightarrow^* (a,b,c), \Leftrightarrow^* (d,b,c))$
for all four-fold tables $\langle a,b,c,d \rangle$*.*

Remark. From the (fuzzy) logic point of view, it means that in all models (data) the formula $\varphi \equiv^* \psi$ is at least so true as the formulae $\varphi \Leftrightarrow^* \psi, \neg\varphi \Leftrightarrow^* \neg\psi$, i.e. deduction rules $\frac{\varphi \Leftrightarrow^* \psi}{\varphi \equiv^* \psi}$, $\frac{\neg\varphi \Leftrightarrow^* \neg\psi}{\varphi \equiv^* \psi}$ are correct.

Example 10 *For the basic double implication*
$\Leftrightarrow_{\oslash} (a,b,c) = \frac{a}{a+b+c}$
the basic equivalence
$\equiv_{\oslash} (a,b,c,d) = \frac{a+d}{a+b+c+d}$
is the result of applying the pseudo-bayesian operator
$x \oplus_B y = \frac{x \cdot (1-y) + (1-x) \cdot y}{1 - x \cdot y}$*:*
$\Leftrightarrow_{\oslash} (a,b,c) \oplus_B \Leftrightarrow_{\oslash} (d,b,c) = \frac{a}{a+b+c} \oplus_B \frac{d}{d+b+c}$
$= \frac{\frac{a}{a+b+c} \cdot (1 - \frac{d}{d+b+c}) + (1 - \frac{a}{a+b+c}) \cdot \frac{d}{d+b+c}}{1 - \frac{a}{a+b+c} \cdot \frac{d}{d+b+c}} = \frac{a+d}{a+b+c+d}.$

6 Conclusion

The results presented in the paper (and the others presented in [9], [11]) indicate a possible role of propositional fuzzy logic in data mining research. Association rules quantified in data can be treated as fuzzy logic formulae in such a way that fuzzy logic instruments could be used for describing and analyzing the structure of knowledge discovered from databases. In particular, some symmetric types of four-fold table quantifiers are related to fuzzy propositional connectives (conjunction and disjunction) in a very natural way, as it is stated in Theorems 1 and 2.

References

[1] Aggraval, R. et al.: Fast Discovery of Association Rules. In Fayyad, V.M. et al.: Advances in Knowledge Discovery and Data Mining. AAAI Press / MIT Press 1996, p.307-328.

[2] Batagelj, V., Bren, M.: Comparing Resemblance Measures. J. of Classification 12 (1995), p. 73-90.

[3] Berka, P., Ivánek, J.: Automated Knowledge Acquisition for PROSPECTOR-like Expert Systems. In Machine Learning. ECML-94 Catania (ed. Bergadano, Raedt). Springer 1994, p.339-342.

[4] Hájek,P., Havránek,T.: Mechanising Hypothesis Formation - Mathematical Foundations for a General Theory. Springer-Verlag, Berlin 1978, 396 p.

[5] Hájek,P., Havránek,T., Chytil M.: Metoda GUHA. Academia, Praha 1983, 314 p. (in Czech)

[6] Hájek, P., Sochorová, A., Zvárová, J.: GUHA for personal computers. Computational Statistics & Data Analysis 19 (1995), p. 149 - 153

[7] Hájek,P.: Metamathematics of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht 1998. 297p.

[8] Hájek, P., Holeňa, M.: Formal Logics of Discovery and Hypothesis Formation by Machine. In Discovery Science (Arikawa,S. and Motoda,H., eds.), Springer-Verlag, Berlin 1998, p.291-302

[9] Ivánek, J.: On the Correspondence between Classes of Implicational and Equivalence Quantifiers. In Principles of Data Mining and Knowledge Discovery. Proc. PKDD'99 Prague (Zytkow,J. and Rauch,J., eds.), Springer-Verlag, Berlin 1999, p.116-124.

[10] Ivánek, J.: Combining Implicational Quantifiers for Equivalence Ones by Fuzzy Operators (preliminary working paper). In: Proc. WUPES'2000 J. Hradec (ed. Vejnarová,J.), Univ. of Economics, Prague, p.102-109.

[11] Ivánek, J.: Construction of Implicational Quantifiers from Fuzzy Implications In: WUPES'2003 Hejnice (ed. Vejnarová,J.), Univ. of Economics, Prague, p.125-132.

[12] Ivánek, J., Stejskal, B.: Automatic Acquisition of Knowledge Base from Data without Expert: ESOD (Expert System from Observational Data). In Proc. COMPSTAT'88 Copenhagen. Physica-Verlag, Heidelberg 1988, p.175-180.

[13] Rauch, J.: Logical Calculi for Knowledge Discovery in Databases. In Principles of Data Mining and Knowledge Discovery, (Komorowski,J. and Zytkow,J., eds.), Springer-Verlag, Berlin 1997, p. 47-57.

[14] Rauch,J.: Classes of Four-Fold Table Quantifiers. In Principles of Data Mining and Knowledge Discovery, (Quafafou,M. and Zytkow,J., eds.), Springer Verlag, Berlin 1998, p. 203-211.

[15] Rauch,J.: 4FT-Miner - popis procedury. Technical Report LISp-98-09, Praha 1999.

[16] Zembowicz,R. - Zytkow,J.: From Contingency Tables to Various Forms of Knowledge in Databases. In Fayyad, U.M. et al.: Advances in Knowledge Discovery and Data Mining. AAAI Press/ The MIT Press 1996, p. 329-349.