Testing hypotheses in exponential families of stochastic processes

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joint work with: Karel Vrbenský

ÚTIA AV ČR

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D. Morales, L. Pardo, I. Vajda.
Some new statistics for testing hypothesis in parametric models.
Journal of Multivariate Analysis, 62: 137 – 168, 1997.

D. Morales, L. Pardo, I. Vajda.
Rényi statistics in directed families of exponential experiments.
Statistics, 34: 151 – 174, 2000.

D. Morales, L. Pardo, M.C. Pardo, I. Vajda.
Rényi statistics for testing composite hypothesis in general exponential models.
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Program on testing hypotheses in exponential models using Rényi divergences

- classical models independent observations coming from the identical distribution with a general density form:

$$p_\theta(x) = \exp(\theta T(x) - \kappa(\theta))$$

- sequences of dependent observations, processes
- random fields

OUR TASK: test of a hypothesis about parameter

$$H_0 : \theta = \theta_0 \quad \text{or} \quad H_0 : \theta \in \Theta_0$$

OUR TEST STATISTICS: derived from $D_a(p_{\hat{\theta}_n} \,|\, p_{\theta_0})$; $a \in \mathbb{R}$

↑ an estimate of parameter (MLE)

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DATA coming from an exponential model

- $\hat{\theta}_t \xrightarrow{n \to \infty} \theta$ a.s.(P_θ)
- $\sqrt{t}(\hat{\theta}_t - \theta)$ has asymptotically $N(0, \ddot{\kappa}^{-1}(\theta))$

$\hat{\theta}_t =$ MLE $\Longrightarrow$ test statistics: $2t\, D_a(p_{\hat{\theta}_t} \,|\, p_{\theta_0})$; $a \in \mathbb{R}$

$H_0 : \theta = \theta_0$

Benefit:

for every given example, i.e. given distribution, θ_0, size of observation we can choose between statistics with respect a to have the best (in some reasonable sense) power

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estimate $\hat{\theta}$ of parameter θ

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DATA: Lévy process X_t given by density

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1. Brownian motion with unknown drift, Poisson process with unknown intensity of jumps

Diffusion processes, Counting processes

Random fields - Ising model

Problems:

in ex 2: manage random process in place of κ:
$\exp\{\theta' T_t(\mathbb{X}_t) - \kappa(\theta) S_t(\mathbb{X}_t)\}$

find asymptotic distribution of Renyi statistics, MLE only numerically)

in generally for continuous time process: observations are discrete - some discrete analogy of density


line 1
line 2
line 3
line 4
line 5
2.5
2
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30
35
40

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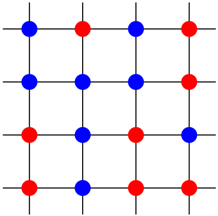

1 Brownian motion with unknown drift, Poisson process with unknown intensity of jumps

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- (2): manage random process $S_t(\mathbb{X}_t)$ from the density: $\exp\{\theta' T_t(\mathbb{X}_t) - \kappa(\theta) S_t(\mathbb{X}_t)\}$
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More examples

1 Brownian motion with unknown drift, Poisson process with unknown intensity of jumps

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X_t ... population size at time t, usually $X_0 = 1$;

each individual gives birth to new individual with rate λ and they behave independently

denote $\theta = \ln(\lambda)$ then

$$\frac{dP_{\theta,t}}{dP_{0,t}} = \exp\{\theta(X_t - 1) - (e^\theta - 1)\int_0^t X_s ds\}$$

we can derive Rényi statistics D_a for $a \in [0, 1]$

we can find their asymptotic behaviour

`Computation for t=9, hypothesis` $H_0 : \lambda = 0.5$

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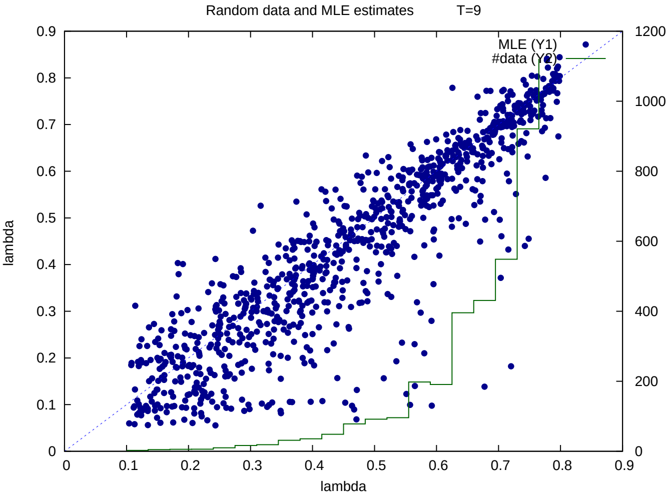
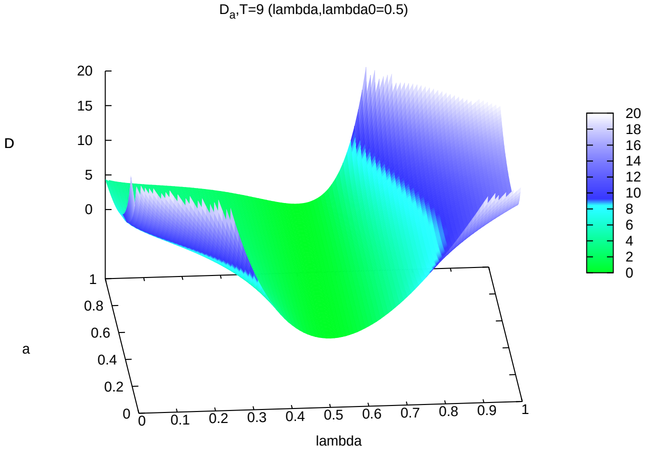
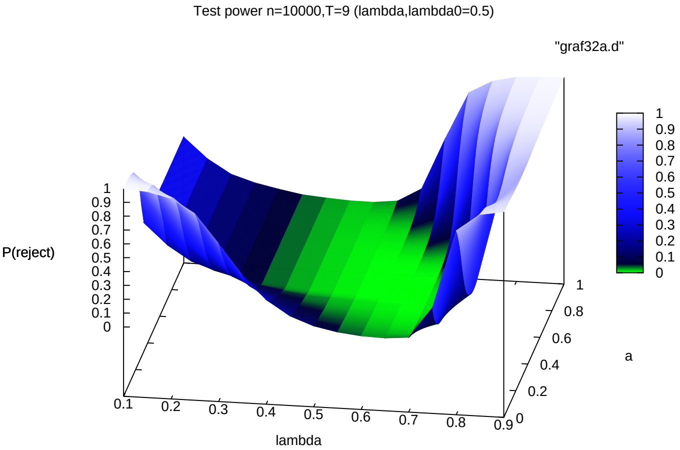

Random data and MLE estimates T=9
MLE (Y1)
#data (Y2)
lambda
lambda
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
200
400
600
800
1000
1200


D_a,T=9 (lambda,lambda0=0.5)
D
20
15
10
5
0
a
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
lambda
20
18
16
14
12
10
8
6
4
2
0


Test power n=10000,T=9 (lambda,lambda0=0.5)
"graf32a.d"
P(reject)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
lambda
0
0.2
0.4
0.6
0.8
1
a
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

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