AGGREGATION OF COMPLEX QUANTITIES

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Summary

We discuss aggregation of complex quantities, namely of intervals distribution functions of copulas and fuzzy quantities. After historical notes, some new aggregation methods for these quantities are proposed.

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1 INTRODUCTION

Aggregation of single real inputs appears almost in any scientific field dealing with observed (measured) real quantities. The theory of aggregation operators is fast developing in last years (recall only edited volumes [1] and [2]). When restricted to the $[0,1]$-domain an aggregation operator A is a non-decreasing $\bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ mapping such that $A(0,\ldots,0)=0$, $A(1,\ldots,1)=1$ and $A(x)=x$ for all $x\in[0,1]$, see, e.g., [11, 12]. Complex quantities describe observed objects with some kind of uncertainty (vagueness). Typical examples of this kind are linguistic quantities (small, medium, large, etc.). Note that we will not focus now on ordinal aspects of these quantities. Therefore, some quantitative model of such complex quantities is needed for our purposes. In the case of linguistic quantities, such a model can be provided in the framework of fuzzy sets [23], rough sets [19], alternative set theory [22], etc., and then we can start to discuss "computing with words" [24, 25]. In classical probability theory a similar problem occurs in calculus with distribution functions. Recall only the convolution of distribution functions (i.e., summation of independent random variables).

There is an argument for the study of general scheme of aggregation operators, exceeding the borders of an actual sort of applications. The "computing with words" can be understood in two ways. The first meaning covers the computation with uncertain input data, where the sequence of computational steps is strictly given by the algorithm, and particular operations are exactly defined. The other meaning is even more liberal - it admits that even particular operations can be specified in several modifications due to the actual model. For example, there exist several ways of a vague instruction formulated as "putting two data together". It can be interpreted as a sum of several random variables (defined by convolution), sum of fuzzy quantities (by the extension principle or another operation), conjunction of vague statements (by fuzzy logical connectives) and numerous other operators. The choice of the proper one of them is sometimes given by the character of the constructed model and also by our subjective interpretation of vague instructions for its processing. Their combination or switching among them represents the qualitatively higher level of management of uncertainty. To be ready to preserve their consistency and unified general character, it is useful to handle more general structure of aggregating operators covering many of the possibilities mentioned above.

The aim of this contribution is to initiate a deeper investigation of aggregation of complex quantities. In the next section, we will discuss the simplest generalization of single real inputs – interval inputs. Aggregation of intervals naturally appears in interval mathematics [16], however, there is still a lot of work to be done. In the third section, we will deal with aggregation of distribution functions. The fourth section is devoted to the aggregation of copulas [17], i.e., special 2-dimensional distribution functions. In section five we will discuss aggregation of fuzzy quantities. To simplify the next considerations, we will aggregate two complex quantities only, the generalization to arbitrary (but finite) number of input complex quantities is mostly straightforward. Note also that whenever involved complex fuzzy quantities are represented as real functions (distribution functions, copulas, fuzzy sets), pointwise aggregation is always possible (and often meaningful). Recall, e.g., the convex combination of two copulas, or the intersection (union) of two fuzzy sets modelled by a t-norm (t-conorm). In such a case, output value in some domain element depends on input values in that domain element only. However, this is not our aim in this paper! We will discuss aggregation of complex quantities which is not provided pointwisely.

What is an aggregation operator $\mathbf{A}$ acting on some kind of complex quantities? In general, we will require the non-decreasingness of $\mathbf{A}$ (with respect to the genuine (partial) order on involved quantities) and preservation of boundaries only. Obviously, for degenerated complex quantities (i.e., when they can be represented by a single real value) we expect that $\mathbf{A}$ is reduced to some (real) aggregation operator A.

2 AGGREGATION OF INTERVALS

Following the ideas of interval mathematics [16], for any continuous aggregation operator A acting on some real interval I, extension of A to (closed) subintervals of I is given by

$$\mathbf{A}([x_1,x_2],[y_1,y_2]) = \{z \in I \mid z = A(x,y), x \in [x_1,x_2], y \in [y_1,y_2]\}.$$

Due to the monotonicity of aggregation operators,

$$\mathbf{A}([x_1,x_2],[y_1,y_2]) = [A(x_1,y_1), A(x_2,y_2)], \qquad (1)$$

see for example [18] for special t-norms on intervals.

Obviously, for a couple of aggregation operator A, B acting on I, such that $A \leq B$, we can define

$$\mathbf{A}([x_1,x_2],[y_1,y_2]) = [A(x_1,y_1), B(x_2,y_2)]. \qquad (2)$$

This approach is applied, e.g., to t-norms on intervals as discussed in [3] (then these t-norms on intervals are called representable).

Inspired by non-representable t-norms as intervals as discussed in [3], we propose (compare also [4]) two next aggregation approaches to intervals (again based on $A \leq B$):

$$\mathbf{A}([x_1,x_2],[y_1,y_2]) = [A(x_1,y_1), \max(B(x_1,y_2), B(y_1,x_2))], \qquad (3)$$

$$\mathbf{A}([x_1,x_2],[y_1,y_2]) = [\min(A(x_1,y_2), A(y_1,x_2)), B(x_2,y_2)]. \qquad (4)$$

Obviously, several other modifications of (3) and (4) can be introduced. As an interesting task for future we put the study of properties of $\mathbf{A}$ based on some prescribed properties of A and B. So, for example, if $A = B$ is a t-norm (on $[0,1]$), then $\mathbf{A}$ given by (3) is an interval-valued t-norm.

3 AGGREGATION OF DISTRIBUTION FUNCTIONS

Classical convolution of distribution functions

$$F_1 * F_2(t) = \int F_1(t-x)\mathrm{d}F_2(x)$$

extends the summation of real quantities based on the product copula Π (i.e., two involved random variables are independent). A similar extension of summation can be introduced for any copula C (expressing the dependence structure of involved random variables), see [17].

An axiomatic approach to aggregation of distribution functions on $[0,\infty]$ was proposed by Šerstnev [20]. His triangle functions can be understood as a kind of t-norms. Moreover, any (left-continuous) t-norm T is linked to a special triangle function τ_T extending the summation of reals,

$$\tau_T(F_1,F_2)(t) = \sup\{T(F_1(t-x), F_2(x)) \mid x \in [0,t]\}. \qquad (5)$$

We can rewrite (5) into

$$\tau_T(F_1,F_2)(t) = \sup\{T(F_1(y), F_2(x)) \mid x,y \in [0,\infty], y+x=t\}. \qquad (6)$$

For arbitrary (left-) continuous aggregation operator A (on $[0,\infty]$) we can define, based on T and inspired by (6), its extension to an aggregation operator $\mathbf{A}_T$ on distribution functions (on $[0,\infty]$) given by

$$\mathbf{A}_T(F_1,F_2)(t) = \sup\{T(F_1(y), F_2(x)) \mid x,y \in [0,\infty], A(y,x)=t\}. \qquad (7)$$

Similarly, for a given continuous t-conorm S [10], we can define

$$\mathbf{A}_S(F_1,F_2)(t) = \inf\{S(F_1(y), F_2(x)) \mid x,y \in [0,\infty], A(y,x)=t\}. \qquad (8)$$

For the strongest t-norm $T_{\mathbf{M}}$, (7) can be transformed (compare [8]) into

$$\mathbf{A}(F_1,F_2) = (A(F_1^{(-1)}, F_2^{(-1)}))^{(-1)}, \qquad (9)$$

where the pseudo-inverse $g^{(-1)} : [c, d] \to [a, b]$ of a non-decreasing function $g : [a, b] \to [c, d]$ is given by [9]

$$g^{(-1)}(t) = \sup\{x \in [a, b] \mid g(x) \leq t\}.$$

Note that we can even relax the requirements on T to be a t-norm in the above considerations (similarly with S). Indeed, T can be chosen to be a t-subnorm [7, 15] which is also an aggregation operator, i.e., T is so called boundary weak t-norm [21].

4 AGGREGATION OF COPULAS

Recall that a copula $C : [0, 1]^2 \to [0, 1]$ is a 2-dimensional distribution function with uniformly (on $[0, 1]$) distributed marginals [17]. Up to pointwise aggregation of copulas (note that in general the class of copulas is closed under convex combinations, i.e., weighted arithmetic means), the only non-pointwise aggregation for copulas was recently proposed in [6]: for a given weight $\lambda \in]0, 1[$, we put

$$\mathbf{W}_\lambda(C_1, C_2) = C_{\lambda R_{C_1} + (1-\lambda) R_{C_2}}, \qquad (10)$$

where $R_C : [0, 1]^2 \to [0, 1]$ is given by

$$R_C(x, y) = \sup\{z \in [0, 1] \mid C(x, z) \leq y\}$$

and $C_R : [0, 1]^2 \to [0, 1]$is given by

$$C_R(x, y) = \inf\{z \in [0, 1] \mid R(x, z) \geq y\}$$

(compare t-norms and residuals of t-norms [10]). Inspired by (10), we open the following problem: for which aggregation operator $A : [0, 1]^2 \to [0, 1]$ is the operator $\mathbf{A}$ acting on functions $H : [0, 1]^2 \to [0, 1]$ and given by

$$\mathbf{A}(H_1, H_2) = C_{A(R_{H_1}, R_{H_2})} \qquad (11)$$

an aggregation operator on copulas? Most probably also in this case weighted means will be the only appropriate aggregation operators solving (11) for copulas. Note however that (11) gives a new method of aggregation of general (binary) aggregation operators.

5 AGGREGATION OF FUZZY QUANTITIES

Arithmetic of fuzzy quantities has been developed in many papers and monographs, recall only [5, 13]. In the case when conjunction (intersection of fuzzy sets) is modelled by $T_{\mathbf{M}}$, arithmetics of fuzzy quantities can be transformed into interval arithmetics of the corresponding α-cuts. Obviously, we can extend this approach to arbitrary (continuous) aggregation operator acting on intervals, see Section 2.

Based on Zadeh's extension principle [23], we can extend aggregation operator $\mathbf{A}_T$ introduced in Section 3 for distribution functions (which are, indeed, special fuzzy quantities) to all fuzzy quantities, using the same formula as in (7). Similarly, (8) can be extended to fuzzy quantities.

In the case of verbally generated fuzzy quantities as proposed in [14], each fuzzy quantity is described by its shape function and by its scale. Following the ideas and notations from [14], we can define an aggregation operator $\mathbf{A}$ on verbally generated fuzzy quantities by

$$\mathbf{A}((\varphi_1, f_1), (\varphi_2, f_2)) = (A(\varphi_1, \varphi_2), B(f_1, f_2)), \qquad (12)$$

where A is an aggregation operator on shapes and B is an aggregation operator on scales.

6 CONCLUSION

We have recalled some known and proposed some new kinds of aggregation operators acting on some class of complex quantities. The main aim of this contribution is to initiate a deeper study of such complex aggregation, whose importance will surely increase in near future.

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