# Shapley Mappings and Values of Coalition Games

Dan Butnariu[1] Tomáš Kroupa[2]

[1]Department of Mathematics
University of Haifa, Israel

[2]Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic

# Coalition Game

- decision-making in the cooperative environment
- coalition game is fully determined by

| | |
|---|---|
| players | $N = \{1, 2, \ldots, n\}$ |
| coalition | $A \in \mathcal{P}$ |
| game | $v : \mathcal{P} \to \mathbb{R}, \quad v(\emptyset) = 0$ |

HOW SHOULD THE PLAYERS DISTRIBUTE GAINS FROM COOPERATION?

| | |
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| Shapley value | linear mapping $\Phi : v \mapsto (\Phi_1(v), \ldots, \Phi_n(v)) \in \mathbb{R}^n$ |
| | - Efficiency |
| | - Null-player condition |
| | - Symmetry |

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# How Payoffs Are to Be Divided?

Shapley (1953) - a normative concept of fair allocation

$$\Phi_i(v) = \sum_{S \in \mathcal{P} \mid i \in S} \frac{(|S| - 1)!\,(n - |S|)!}{n!}(v(S) - v(S \setminus \{i\}))$$

# Example

- The player 1 (owner of business) does not work but provides the crucial capital: without him no gains can be obtained.
- Other players $2, \ldots, 11$ (workers) each contributing the amount \$10 000 to the total profit.

Model:

$N = \{1, \ldots, 11\}$

$$v(A) = \begin{cases} 0, & 1 \notin A, \\ 10\,000 \cdot (|A| - 1), & 1 \in A. \end{cases}$$

Shapley value $\Phi_1(v) = 50\,000$, $\Phi_i(v) = 5\,000$ for $i > 1$.

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# Classes of Coalition Games

- $N$ finite, $A \subseteq N$, $v : \mathcal{P} \to \mathbb{R}$ (Shapley; 1953)
- $N$ infinite, $A$ is a measurable subset of $N$, game $v$ is a non-atomic measure (Aumann, Shapley; 1974)
- $N$ finite,
  $A$ is a (fuzzy) coalition $A = (A(1), \ldots, A(n)) \in [0,1]^N$,
  game is a function $[0,1]^N \to \mathbb{R}$ (Aubin; 1974)

Correspondence:

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| coalition $A \subseteq N$ | $\longleftrightarrow$ | corner of the unit cube $[0,1]^n$ |
| fuzzy coalition $A$ | $\longleftrightarrow$ | point of the unit cube $[0,1]^n$ |

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# The Model

IDEA: Inspect the structure of coalitions level-by-level

players on the same level $A_t = \{i \in N \mid A(i) = t\}, \quad t \in [0,1]$

weight function $\psi : [0,1] \to \mathbb{R}$

$(\psi(t) = 0 \Leftrightarrow t = 0)$ and $(\psi(1) = 1)$

class of games studied

$$\mathcal{G}[\psi] = \left\{ v : [0,1]^N \to \mathbb{R} \quad \middle| \quad v(A) = \sum_{t \in [0,1]} \psi(t) v(A_t) \right\}$$

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# Motivating Example

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| $N$ | ... | set of investors, each $i \in N$ having capital $c_i$ |
| $A(i)$ | ... | ratio of $i$'s investment in $A$ to $c_i$ |
| | | measure of the risk |
| $g : \mathbb{R} \to \mathbb{R}$ | ... | expected return function satisfying |
| | | $g(tq) = tg(q), \; t \in [0, 1]$ |
| $\chi : [0, 1] \to \mathbb{R}$ | ... | risk rewarding function satisfying $\chi(1) = 1$ |

The player invests $q$ (fraction $t$ of his capital)

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the player gets $\chi(t)g(q)$ instead of $g(q)$

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# Motivating Example (ctnd.)

Each fuzzy coalition $A$ is assigned the total expected revenue of its level sets weighted with the rewards for each level of risk:

$$v(A) = \sum_{t \in [0,1]} \chi(t)\, g\Big(t \sum_{i \in A_t} c_i\Big) = \sum_{t \in [0,1]} \chi(t) t\, g\Big(\sum_{i \in A_t} c_i\Big)$$
$$= \sum_{t \in [0,1]} \psi(t) v(A_t)$$

▸ Check

Crucial questions for player $i$:

1. What is $i$'s expected return of investing a share $A(i)$ of his capital into the coalition $A$?
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# Shapley Mapping: Axiomatic Approach

## Definition

A Shapley mapping is a linear mapping $\Phi : \mathcal{G}[\psi] \to (\mathbb{R}^N)^{[0,1]^N}$ such that for any $v \in \mathcal{G}[\psi]$ and any $A \in [0,1]^N$:

**Efficiency** For every $v$-carrier $B \in [0,1]^N$ of $A$:

$$\sum_{i \in N : B(i) > 0} \Phi_i(v)(A) = v(B)$$

**Non-Member** If $A(j) = 0$, then $\Phi_j(v)(A) = 0$

**Symmetry** If $\pi$ is a permutation of $N$, then

$$\Phi_{\pi i}(\pi v)(\pi A) = \Phi_i(v)(A)$$

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# Shapley Mapping: Constructive Approach

## Theorem

*There exists a unique Shapley mapping*

$$\Phi : v \mapsto (\Phi_1(v)(A), \ldots, \Phi_n(v)(A))$$

*and*

$$\Phi_i(v)(A) = \begin{cases} \psi(r) \sum_{S \in \mathcal{P}_i(A_r)} \frac{(|S|-1)!(|A_r|-|S|)!}{|A_r|!}(v(S) - v(S \setminus \{i\})), & \text{if } A(i) = r \\ 0, & \text{otherwise,} \end{cases}$$

*where*

$$\mathcal{P}_i(A_r) = \{R \subseteq N | i \in R \text{ and } R \subseteq A_r\}.$$

WHAT IS THE EXPECTED TOTAL ALLOCATION $\Phi_i(v)$ OF PLAYER $i$ IN THE COOPERATIVE PROCESS?

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# Cumulative Value

## Definition

The cumulative value of player $i$ is defined as

$$\Phi_i(v) = \int_{[0,1]^N} \Phi_i(v)(A)\, dA$$

## Theorem

*If the weight function $\psi$ is bounded and Lebesgue integrable, then the cumulative value $\Phi_i(v)$ is well defined and*

$$\Phi_i(v) = v(\{i\}) \int_0^1 \psi(t)\, dt$$

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# Game from $\mathcal{G}[\psi]$ and Outside $\mathcal{G}[\psi]$

## Example

- $N = \{1, 2\}$, $\psi(t) = t$
- $v$ defined by $v(\{1\}) = 1$, $v(\{2\}) = 1$, $v(\{1, 2\}) = 3$ determines a game that is not continuous at $(1, 1)$
- Shapley mapping $\Phi_i(v)(A) = A(i)$, cumulative value $\Phi_i(v) = 0.5$

## Example

- $N = \{1, 2\}$
- $v(A) = \max_{i \in N} A(i), \quad v \notin \mathcal{G}[\psi]$
- Aubin's value is the subgradient

$$\partial v(1, 1) = \{x \in [0, 1]^n \mid \langle x, d \rangle \leq v'_d(1, 1), \forall d \in \mathbb{R}^2\}$$

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