COMPOSITIONAL MODELS

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Oligodimensional distributions

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$$P_1(x_{K_1}), P_2(x_{K_2}), \ldots, P_n(x_{K_n})$$

Definition of the operator of composition

For 2 distributions:

$P(x_K), Q(x_L)$

their *composition* is defined by the expression

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More precisely:

$$P \triangleright Q = \begin{cases} \frac{P(x_K)Q(x_L)}{Q(x_{K\cap L})} & \text{if } P(x_{K \cap L}) \ll Q(x_{K \cap L}), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

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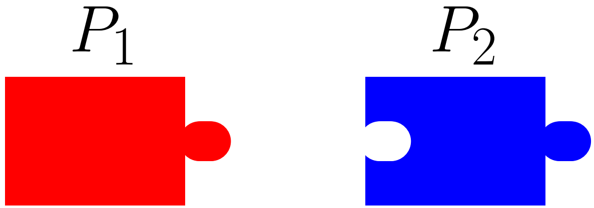
- $X_{K \setminus L} \perp\!\!\!\perp X_{L \setminus K} | X_{K \cap L} \; [P \triangleright Q]$
- if P and Q are consistent then $P \triangleright Q$ is a maximum entropy extension of both P and Q.

Composition of distributions



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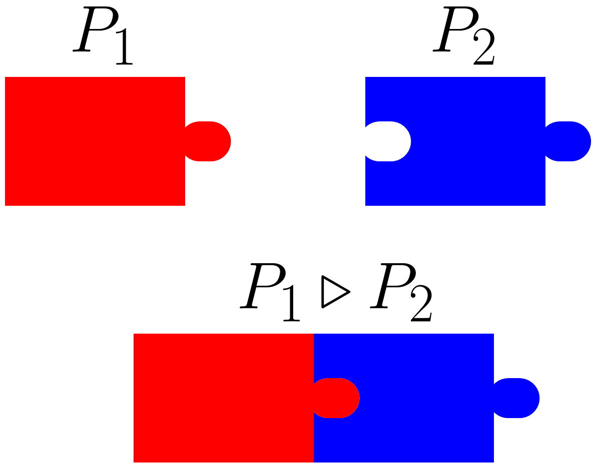

Generating sequence

System $\{P_j(x_{K_j}) \ : j = 1, 2, \ldots, m\}$.

Compositional model is defined by a *generating sequence* $(m = 2)$:

P_1, P_2 define distribution:

$P_1(x_{K_1}) \triangleright P_2(x_{K_2})$

which is defined for variables $X_{(K_1 \cup K_2)}$

Generating sequence

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Compositional model is defined by a *generating sequence* $(m = 3)$:

P_1, P_2, P_3 define distribution:

$$P_1(x_{K_1}) \triangleright P_2(x_{K_2}) \triangleright P_3(x_{K_3})$$
$$= (P_1 \triangleright P_2) \triangleright P_3$$

which is defined for variables $X_{(K_1 \cup K_2 \cup K_3)}$

Generating sequence

System $\{P_j(x_{K_j}) : j = 1, 2, \ldots, m\}$.

Compositional model is defined by a *generating sequence* $(m = 4)$:

P_1, P_2, P_3, P_4 define distribution:

$$P_1(x_{K_1}) \triangleright P_2(x_{K_2}) \triangleright P_3(x_{K_3}) \triangleright P_4(x_{K_4})$$
$$= ((P_1 \triangleright P_2) \triangleright P_3) \triangleright P_4$$

which is defined for variables $X_{(K_1 \cup K_2 \cup K_3 \cup K_4)}$

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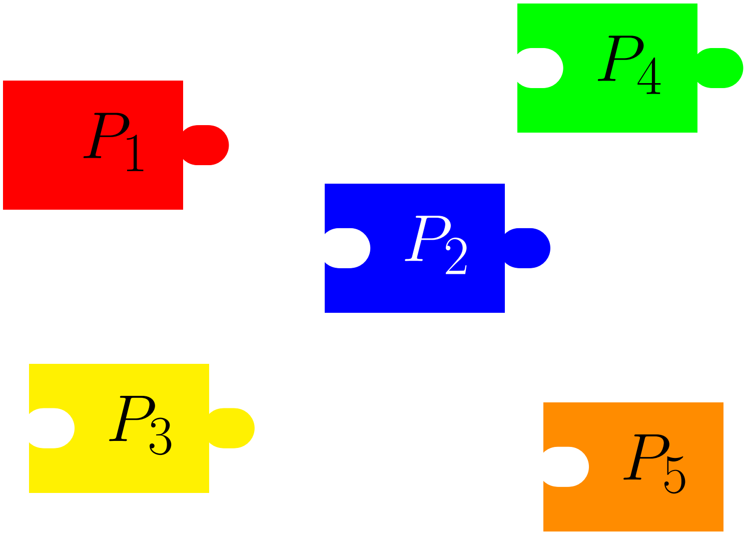
Compositional model is defined by a *generating sequence*:

$P_1, \ldots, P_m$ define distribution:

$$P_1(x_{K_1}) \triangleright P_2(x_{K_2}) \triangleright \ldots \triangleright P_m(x_{K_m})$$
$$= (\ldots(P_1 \triangleright P_2) \triangleright \ldots) \triangleright P_m)$$

which is defined for variables $X_{(K_1 \cup \ldots \cup K_m)}$

Composition of oligodimensional distributions



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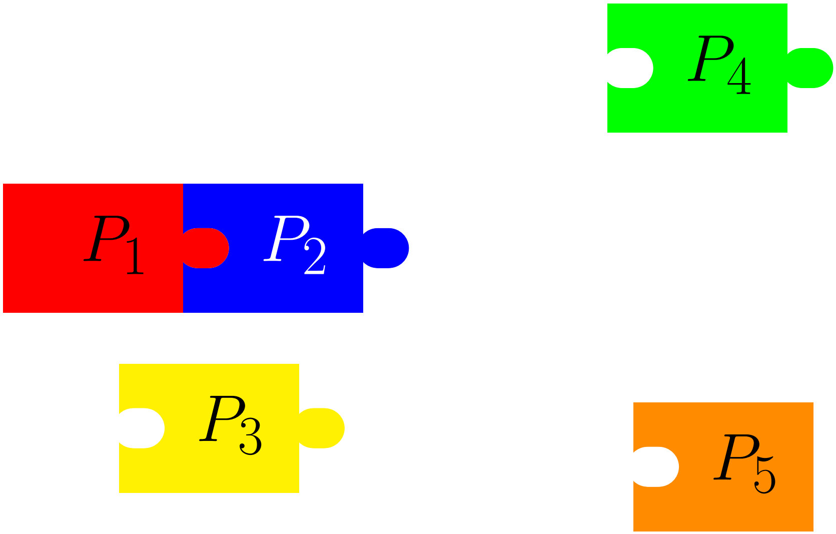

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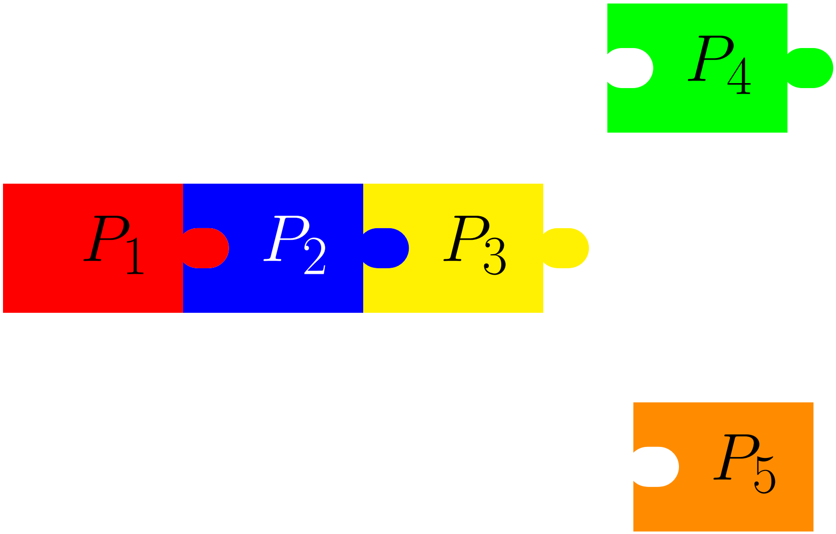

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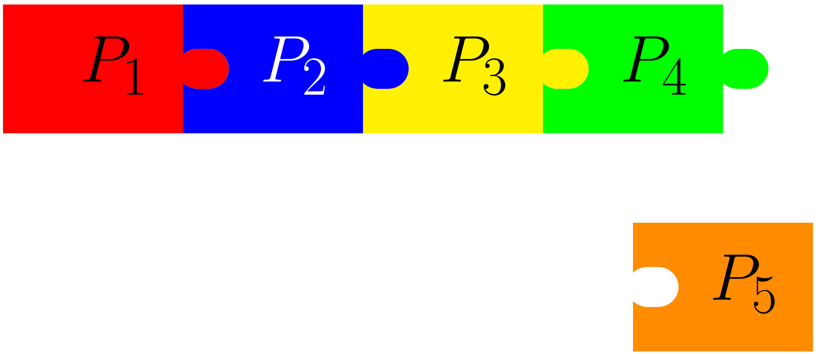

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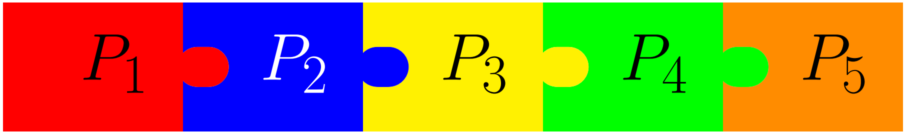

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Denote $L = K_j \cap (K_1 \cup \ldots \cup Kj - 1)$,

$$(P_1 \triangleright \ldots \triangleright P_{j-1}) \triangleright P_j = \frac{(P_1 \triangleright \ldots \triangleright P_{j-1}) \cdot P_j}{(P_j)^{\downarrow L}}$$

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To define multidimensional distribution it is enough to determine a sequence of oligodimensional distributions

GENERATING SEQUENCE

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GENERATING SEQUENCE

advantageous are

PERFECT SEQUENCES

they define the same distriibution for both operators

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$$\vdots$$

$$P_1 \triangleright P_2 \triangleright \ldots \triangleright P_n = P_1 \triangleleft P_2 \triangleleft \ldots \triangleleft P_n$$

Basic properties of perfect sequences

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- $P_1, \ldots, P_m$ is perfect *iff* all the distributions P_j are marginal to $P_1 \triangleright \ldots \triangleright P_m$

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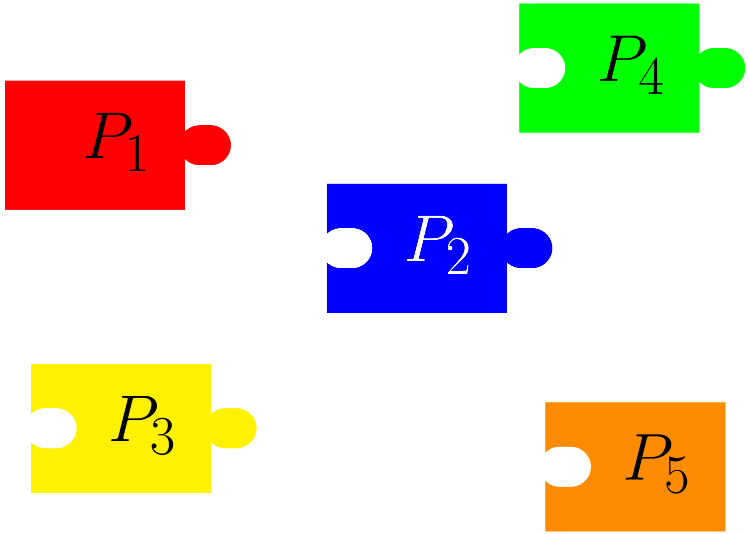
- $P_1, \ldots, P_m$ is perfect *iff* all the distributions P_j are marginal to $P_1 \triangleright \ldots \triangleright P_m$
- The class Bayesian network models is equivalent to the class of perfect sequence models.

Composition of oligodimensional distributions



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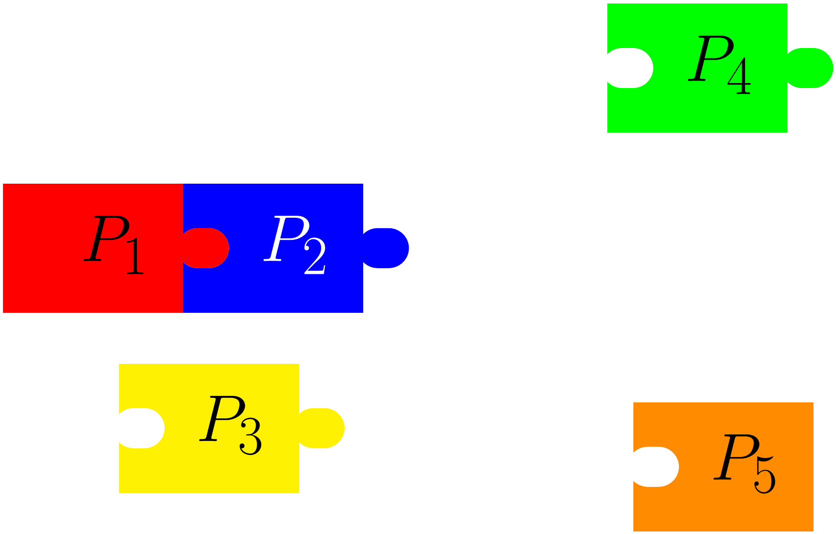

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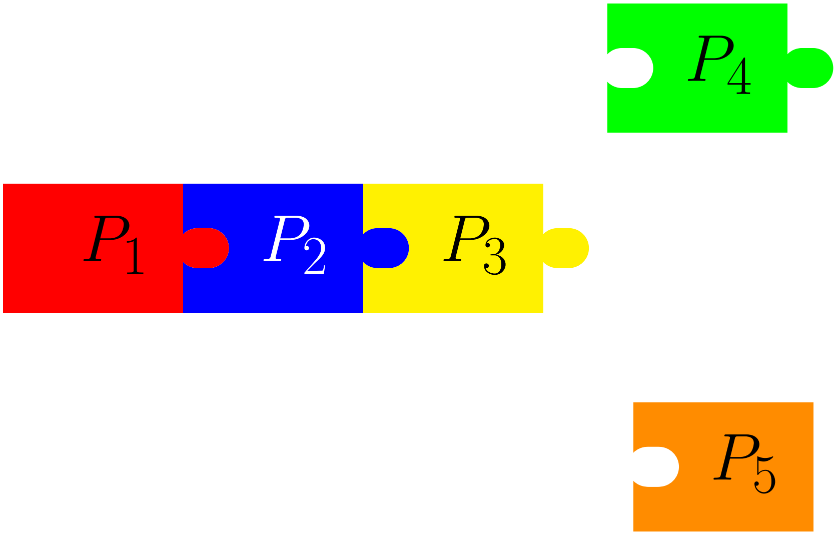

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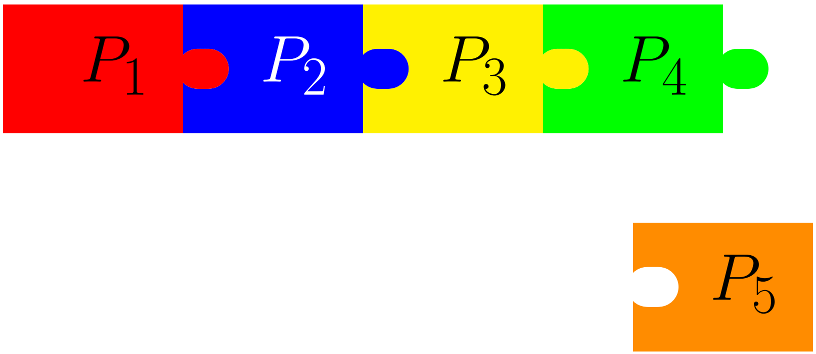

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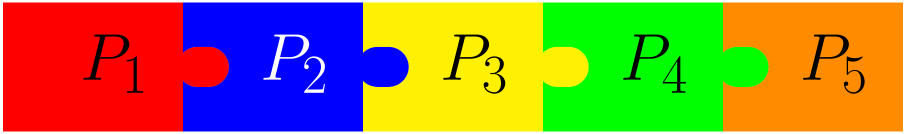
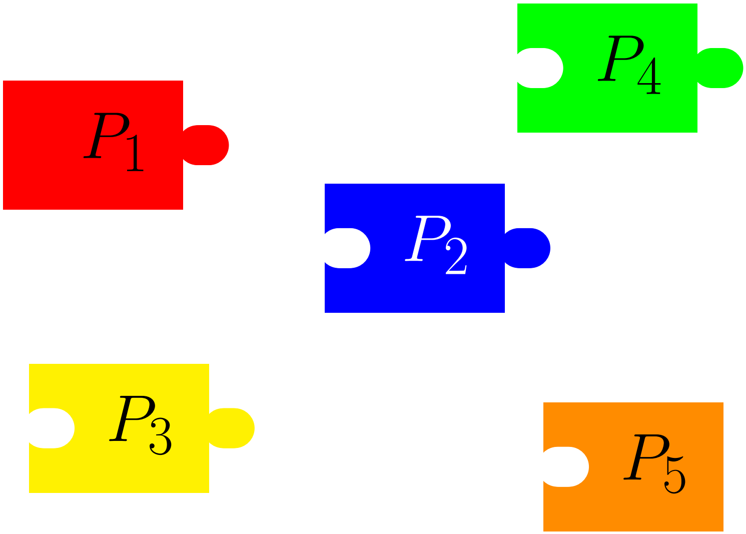


Perfect sequence

Composition of oligodimensional distributions



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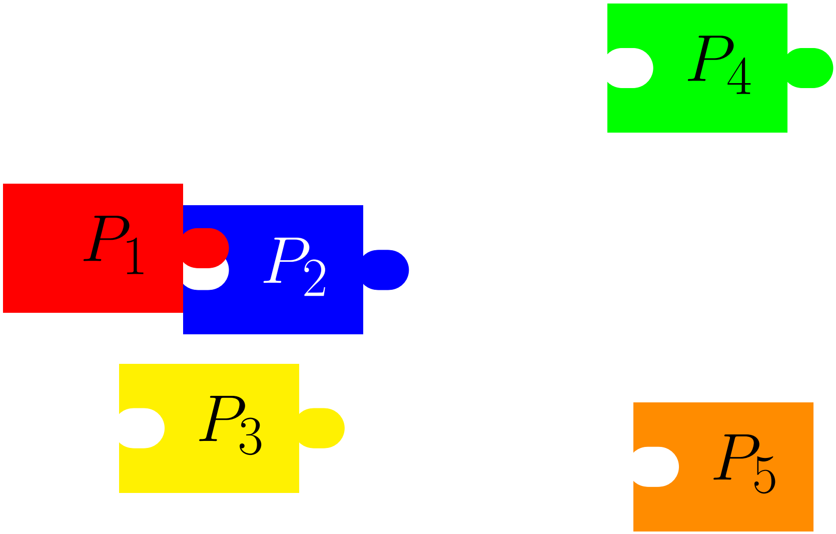

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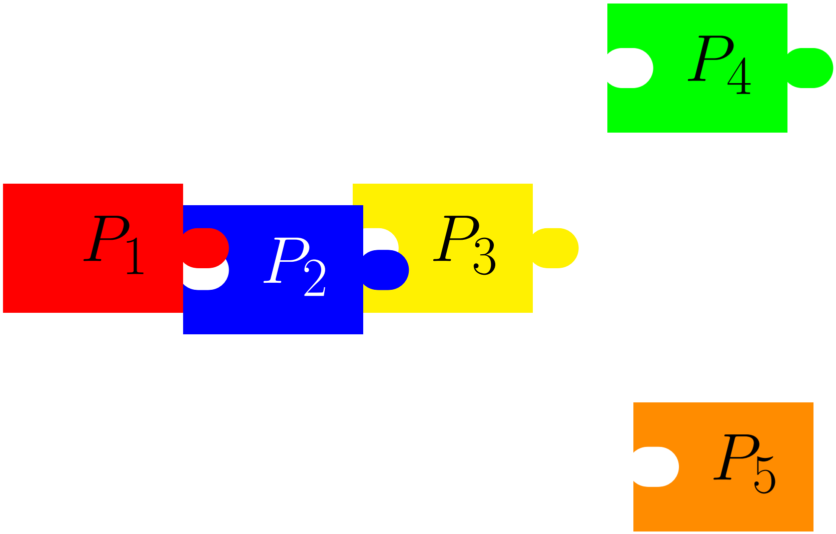

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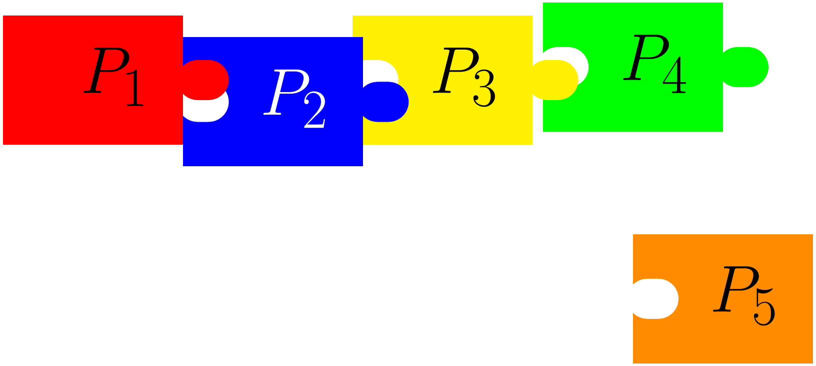

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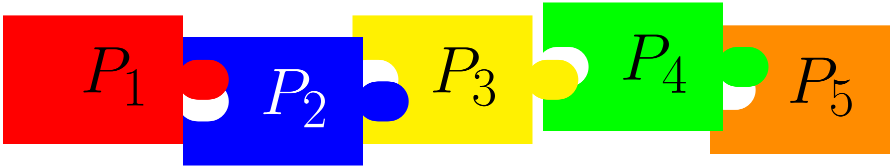

Non-perfect sequence

Comments on equivalence of BN and PS models

- Simple algorithm transforming these types of representation each to other;
- Transformation PS $\longrightarrow$ BN is local;
- Transformation BN $\longrightarrow$ PS is not local (!);
- The difference arises from the fact that in PS models some of marginal distriobutions are explicitely expressed - their computation from BN is sometimes computationally expensive;
- Consequence: simplification of some computational procedures.

What have been achieved in the field of PS models?

- Theoretical apparatus
- Persegrams
- Decomposition
- Efficient marginalization procedure

Comparison of BN and PS models

Shachter's procedure of node deletion and edge reversal

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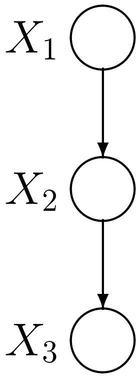

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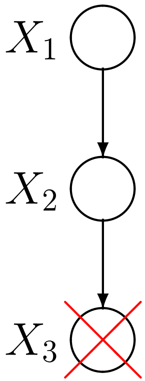

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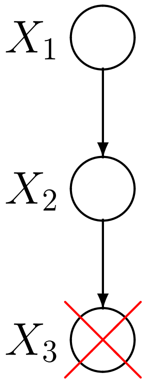
Bayesian network | **Compositional model**

terminal node deletion



$$Q_1(X_1, X_2) \triangleright Q_2(X_2, X_3)$$

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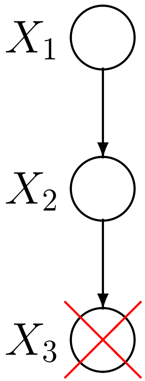

Compositional model

deletion of a variable appearing among the arguments of only one distribution

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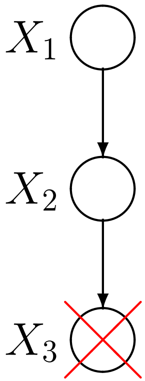

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Decomposition of PS models

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$$P_1(x_1, x_2), P_2(x_1, x_3), P_3(x_3, x_4, x_7), P_4(x_2, x_5), P_5(x_3, x_6), P_6(x_5, x_6, x_9),$$
$$P_7(x_6, x_7, x_{11}), P_8(x_5, x_8), P_9(x_6, x_{10}), P_{10}(x_8, x_9, x_{10}, x_{12})$$

Decomposition of PS models

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$P_1, P_2, P_4, P_5, P_6, P_8, P_9, P_{10}$

P_3, P_5, P_7

Decomposition of PS models

$$P_1(x_1, x_2), P_2(x_1, x_3), P_3(x_3, x_4, x_7), P_4(x_2, x_5), P_5(x_3, x_6), P_6(x_5, x_6, x_9),$$
$$P_7(x_6, x_7, x_{11}), P_8(x_5, x_8), P_9(x_6, x_{10}), P_{10}(x_8, x_9, x_{10}, x_{12})$$

$P_1, P_2, P_4, P_5, P_6, P_8, P_9, P_{10}$

P_3, P_5, P_7

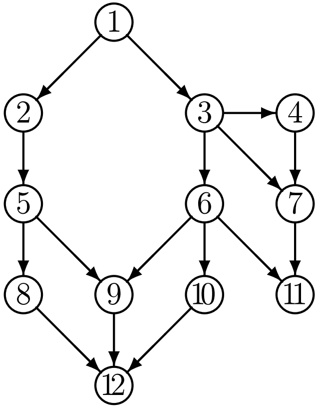
$P_1, P_2, P_4, P_5, P_6,$

P_6, P_8, P_9, P_{10}

$P_3,$

$P_3^{\{(3,7)\}}, P_5, P_7$

Decomposition of Bayesian networks



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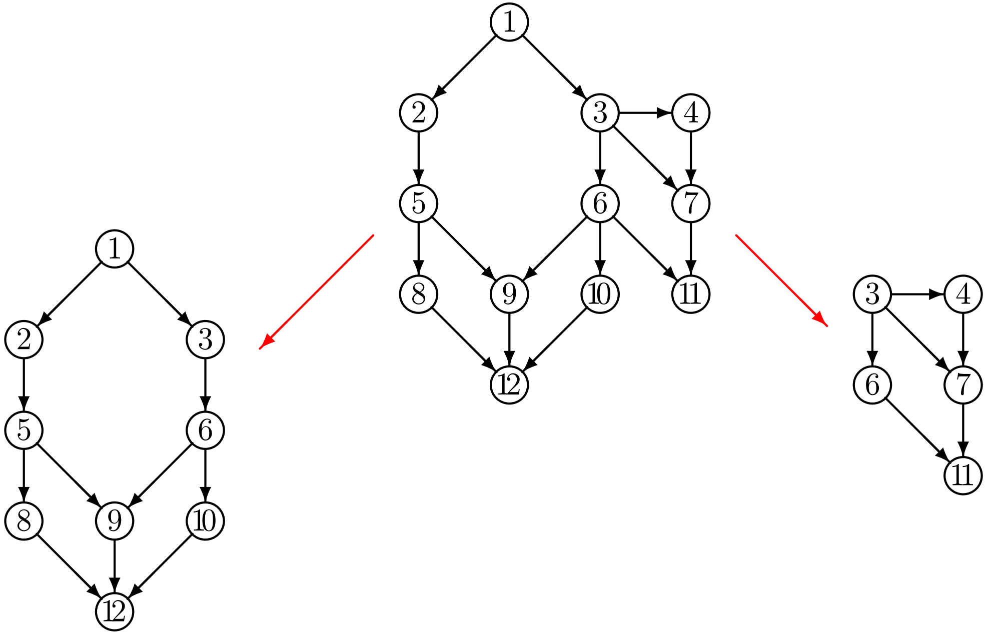

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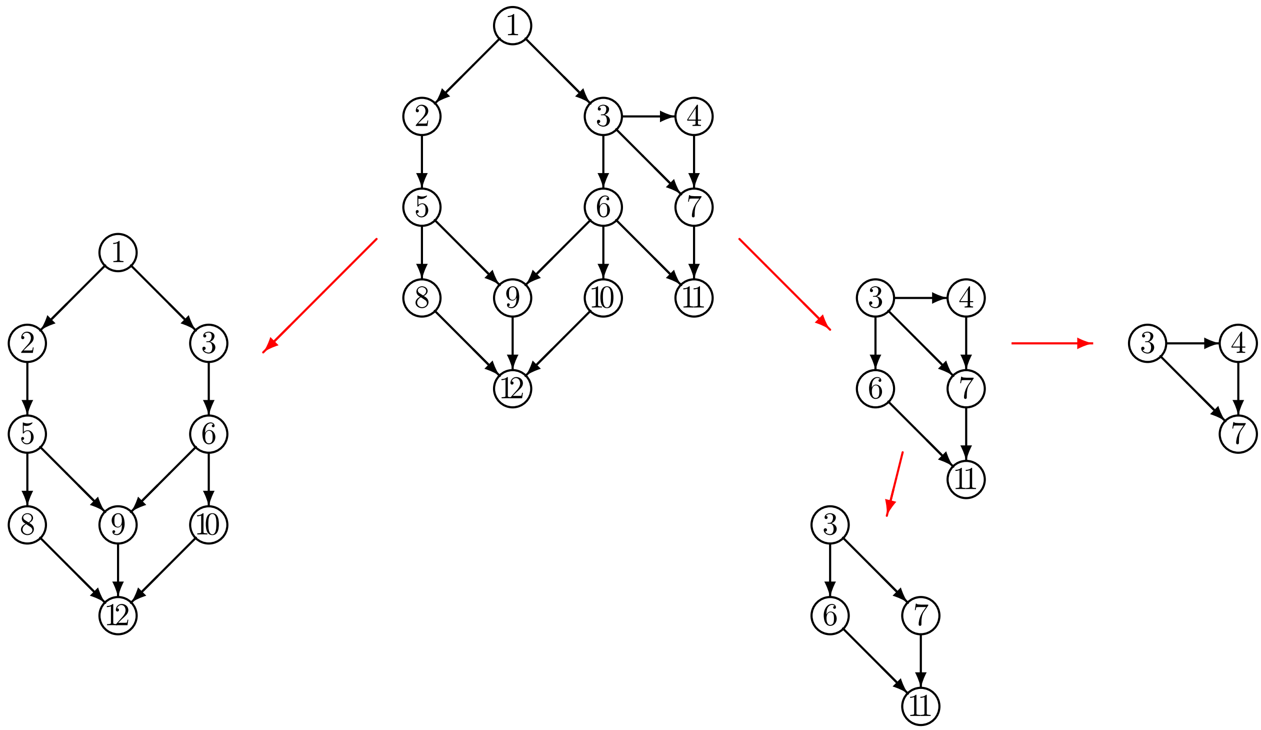

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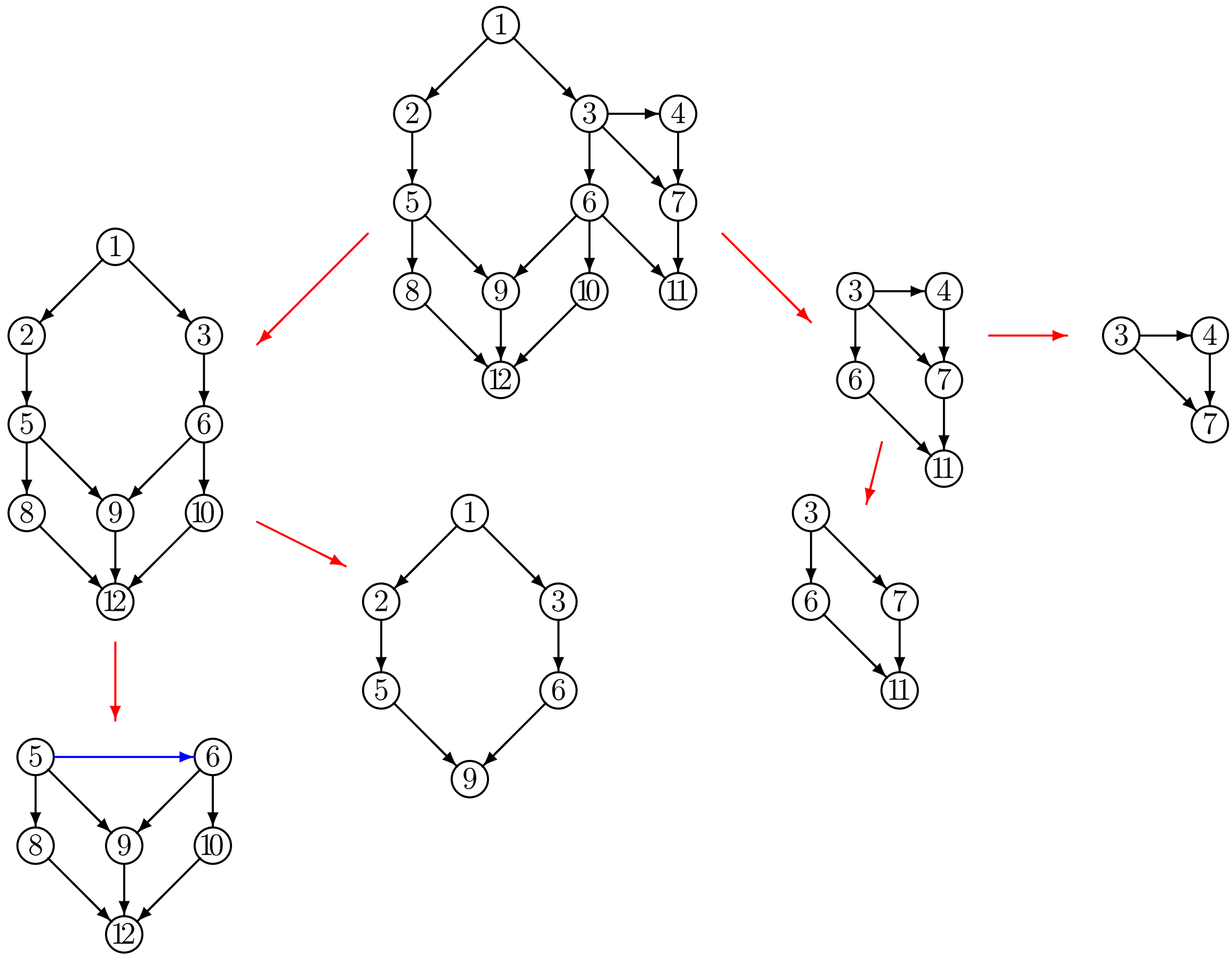

Concluding remark

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Based on Albert Perez' ideas of dependence structure siplification approximation we have developed an approach for multidimensional distributions modelling, which is an alternative to Bayesian networks and which is from some points of view superioir to them.