Adaptive Choice of Scaling Parameter in Derivative-Free Local Filters

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System description

Stochastic dynamic system

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \mathbf{w}_k,\ \ k = 0, 1, 2, \ldots$$
$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k,\ \ k = 0, 1, 2, \ldots$$

is considered, where

- $\mathbf{x}_k$ is immeasurable state of the system,
- $\mathbf{z}_k$ is the measurement,
- $\mathbf{F}_k$ is the known matrix, $\mathbf{h}_k(\cdot)$ is the known vector function,
- $\mathbf{w}_k$ and $\mathbf{v}_k$ are the state and measurement noises with known pdf's $p(\mathbf{w}_k) = \mathcal{N}\{\mathbf{w}_k : \mathbf{0}, \mathbf{Q}_k\}$ and $p(\mathbf{v}_k) = \mathcal{N}\{\mathbf{v}_k : \mathbf{0}, \mathbf{R}_k\}$, respectively,
- the noises are mutually independent and independent of the initial state $\mathbf{x}_0$ with $p(\mathbf{x}_0) = \mathcal{N}\{\mathbf{x}_0 : \bar{\mathbf{x}}_0, \mathbf{P}_0\}$.

State estimation - filtering

The goal of the filtering is to find a pdf of the state $\mathbf{x}_k$ conditioned by the measurements $\mathbf{z}^k = [\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k]$, i.e. $p(\mathbf{x}_k|\mathbf{z}^k)$, or the conditional mean $\hat{\mathbf{x}}_{k|k} = E[\mathbf{x}_k|\mathbf{z}^k]$ and cov. matrix $\mathbf{P}_{k|k} = \mathrm{cov}[\mathbf{x}_k|\mathbf{z}^k]$.

Approximate nonlinear filters based on Bayesian approach

- global filters
  - analytical - Gaussian sum filter
  - simulation - particle filters
  - numerical - point-mass method
- local filters
  - standard (1970) - extended Kalman Filter, second order filter
  - derivative-free (2000) - unscented Kalman filter, divided difference filters

Unified framework for local filters

- Set $k = 0$, $\hat{\mathbf{x}}_{0|-1} = \bar{\mathbf{x}}_0$, and $\mathbf{P}_{0|-1} = \mathbf{P}_0$.
- Filtering estimate is given by

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{xz,k|k-1}\mathbf{P}_{z,k|k-1}^{-1}(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}),$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{xz,k|k-1}\mathbf{P}_{z,k|k-1}^{-1}\mathbf{P}_{xz,k|k-1}^{T},$$

  where

$$\hat{\mathbf{z}}_{k|k-1} = E[\mathbf{z}_k|\mathbf{z}^{k-1}] = E[\mathbf{h}_k(\mathbf{x}_k)|\mathbf{z}^{k-1}],$$

$$\mathbf{P}_{z,k|k-1} = E[(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})^T|\mathbf{z}^{k-1}],$$

$$\mathbf{P}_{xz,k|k-1} = E[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})^T|\mathbf{z}^{k-1}].$$

- Predictive estimate is given by

$$\hat{\mathbf{x}}_{k+1|k} = E[\mathbf{x}_{k+1}|\mathbf{z}^k] = \mathbf{F}_k\hat{\mathbf{x}}_{k|k},$$

$$\mathbf{P}_{k+1|k} = \mathrm{cov}[\mathbf{x}_{k+1}|\mathbf{z}^k] = \mathbf{F}_k\mathbf{P}_{k|k}\mathbf{F}_k^T + \mathbf{Q}_k.$$

Measurement predictive statistics using unscented transformation - scalar variables

Set of weighted predictive σ-points

$$\mathcal{X}_{0,k|k-1} = \hat{x}_{k|k-1}, \qquad \mathcal{X}_{1,2,k|k-1} = \hat{x}_{k|k-1} \pm \sqrt{(1+\kappa)P_{k|k-1}},$$

$$\mathcal{W}_0 = \frac{\kappa}{1+\kappa}, \qquad \mathcal{W}_{1,2} = \frac{1}{2(1+\kappa)},$$

is transformed through the nonlinear function

$$\mathcal{Z}_{i,k|k-1} = h_k(\mathcal{X}_{i,k|k-1}), \; i = 0,1,2,$$

and the resulting statistics are given by

$$\hat{z}_{k|k-1} = \sum_{i=0}^{2} \mathcal{W}_i \mathcal{Z}_{i,k|k-1},$$

$$P_{z,k|k-1} = \sum_{i=0}^{2} \mathcal{W}_i (\mathcal{Z}_{i,k|k-1} - \hat{z}_{k|k-1})(\mathcal{Z}_{i,k|k-1} - \hat{z}_{k|k-1})^T,$$

$$P_{xz,k|k-1} = \sum_{i=0}^{2} \mathcal{W}_i (\mathcal{X}_{i,k|k-1} - \hat{x}_{k|k-1})(\mathcal{Z}_{i,k|k-1} - \hat{z}_{k|k-1})^T.$$

UKF and recommended setting of scaling parameter

- Utilisation of the unscented transformation in the local filter framework leads to the unscented Kalman filter (UKF).
- Design of the UKF is conditioned by specification of the scaling parameter κ.
- The scaling parameter affects spreading of the σ-points in the state space. It thus affects estimation performance of the UKF.
- Standard choice is $\kappa = 3 - n_x$ if $n_x < 3$, else $\kappa = 0$.

System description - bearings only tracking

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.9 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}_k + \mathbf{w}_k,$$

$$z_k = \tan^{-1}\left(\frac{x_{2,k} - \sin(k)}{x_{1,k} - \cos(k)}\right) + v_k,$$

where $k = 0, 1, \ldots, 500$, $p(\mathbf{x}_0) = \mathcal{N}\{\mathbf{x}_0 : [20, 5]^T, 0.1\mathbf{I}\}$,
$\mathbf{Q}_k = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{bmatrix}$, $R_k = 0.025$, $\forall k$.

MSE for UKF's with different choices of κ

	$\boldsymbol{\kappa = 0}$	$\boldsymbol{\kappa = 1}$	$\kappa = 2$	$\kappa = 4$
MSE	23.66	14.35	9.09	4.79

Standard choice of scaling parameter

- The scaling parameter is chosen prior to estimation experiment.
- The parameter does not reflect the particular system description (except for the state dimension) as well as the particular working point.

Goal of the paper

The goal is to propose a technique for adaptive setting of the scaling parameter.

Measurement predictive statistics

- The measurement predictive statistics $\hat{\mathbf{z}}_{k|k-1}$, $\mathbf{P}_{z,k|k-1}$, and $\mathbf{P}_{xz,k|k-1}$ depend on the scaling parameter κ, i.e.

$$\hat{\mathbf{z}}_{k|k-1} = \hat{\mathbf{z}}_{k|k-1}(\kappa), \mathbf{P}_{(x)z,k|k-1} = \mathbf{P}_{(x)z,k|k-1}(\kappa).$$

- The approximate likelihood function thus depends on κ as well, i.e.

$$\hat{p}(\mathbf{z}_k|\mathbf{z}^{k-1}, \kappa) = \mathcal{N}\{\mathbf{z}_k : \hat{\mathbf{z}}_{k|k-1}(\kappa), \mathbf{P}_{z,k|k-1}(\kappa)\}.$$

Technique for adaptive choice of parameter

The proposed technique is based on maximisation of the approx. likelihood function. The scaling parameter is determined as

$$\hat{\kappa}_k = \arg\max_{\kappa} \hat{p}(\mathbf{z}_k|\mathbf{z}^{k-1}, \kappa).$$

Unified framework for local filters with adaptive choice of scaling parameter

- Set $k = 0$, $\hat{\mathbf{x}}_{0|-1} = \bar{\mathbf{x}}_0$, and $\mathbf{P}_{0|-1} = \mathbf{P}_0$.
- Compute the scaling parameter $\hat{\kappa}_k$ maximising the approximate likelihood function.
- Filtering estimate is given by

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{xz,k|k-1}\mathbf{P}_{z,k|k-1}^{-1}(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}),$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{xz,k|k-1}\mathbf{P}_{z,k|k-1}^{-1}\mathbf{P}_{xz,k|k-1}^T,$$

  where $\hat{\mathbf{z}}_{k|k-1}$, $\mathbf{P}_{z,k|k-1}$, and $\mathbf{P}_{xz,k|k-1}$ are computed using $\hat{\kappa}_k$.
- Predictive estimate is given by

$$\hat{\mathbf{x}}_{k+1|k} = E[\mathbf{x}_{k+1}|\mathbf{z}^k] = \mathbf{F}_k\hat{\mathbf{x}}_{k|k},$$
$$\mathbf{P}_{k+1|k} = \mathrm{cov}[\mathbf{x}_{k+1}|\mathbf{z}^k] = \mathbf{F}_k\mathbf{P}_{k|k}\mathbf{F}_k^T + \mathbf{Q}_k.$$

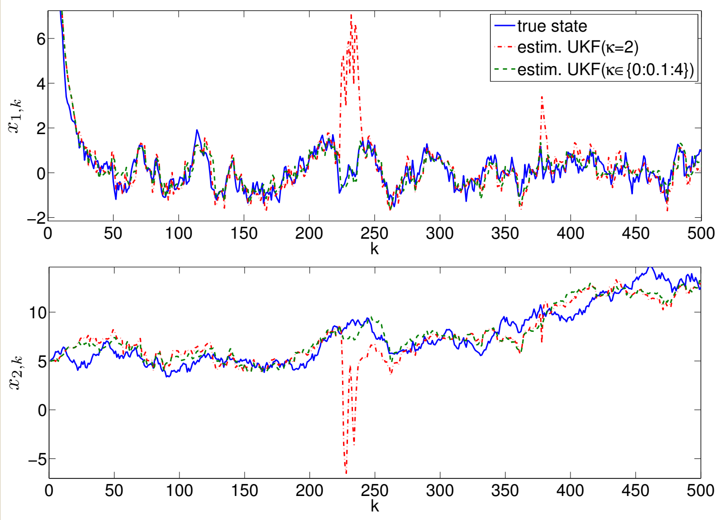
Numerical illustration - specification

- The UKF's with the fixed scaling parameter are compared with UKF's with the adaptively chosen parameter.
- Maximisation is performed using the grid method (the likelihood function is evaluated in several grid points).
- Used notation $\kappa \in \{\kappa_{min} : \kappa_{step} : \kappa_{max}\}$ means the points are equally spread between κ_{min} and κ_{max} with increment κ_{step}.

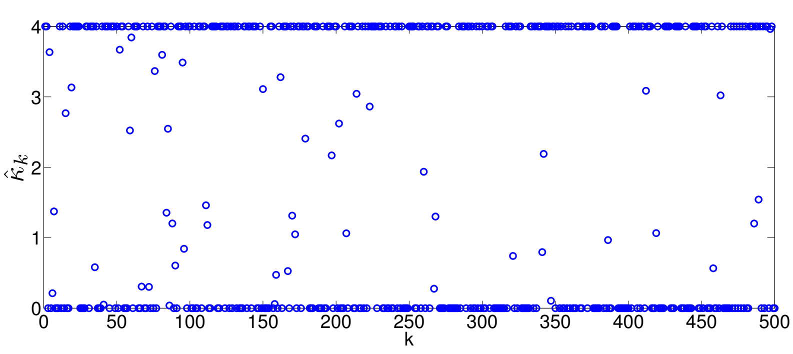
MSE for UKF's with fixed and adaptive choices of κ

	$\kappa = 0$	$\kappa = 4$	$\kappa \in \{0:0.1:4\}$	$\kappa \in \{0:4:4\}$
MSE	23.66	4.79	2.69	2.76
time	0.0016		0.0330	0.0030

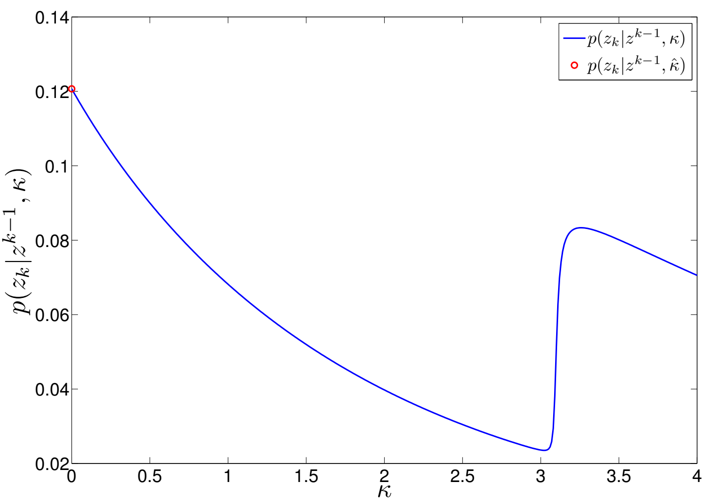
Example of true and estimated state trajectories



Values of scaling parameter with maximal likelihood



Example of likelihood function



System description

$$x_{k+1} = (1 - 0.05\Delta T)x_k + 0.04\Delta T x_k^2 + w_k,$$
$$z_k = x_k^2 + x_k^3 + v_k,$$

where $k = 0, 1, \ldots, 150$, $\Delta T = 0.01$, $p(x_0) = \mathcal{N}\{x_0 : 2.3, 0.01\}$, $Q_k = 0.5$, $R_k = 0.09$, $\forall k$.

MSE for UKF's with fixed and adaptive choices of κ

	$\kappa = 0$	$\kappa = 3$	$\kappa = 4$	$\kappa \in \{0:0.1:4\}$
MSE	0.77	0.11	0.12	0.08

Concluding remarks

- Impact of the scaling parameter on the derivative-free filter estimation performance was discussed.
- The novel technique for adaptive setting of the scaling parameter was designed.
- The technique was illustrated by means of the UKF using numerical examples.
- The technique is easily applicable to all local filters with one or more scaling parameters.