Linear Uniform State-Space Model of Traffic Flow and its Estimation

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Presentation layout

- Motivation
- Model description
- Bayesian estimation
- Introduction to the traffic problem
- Examples
- Conclusions

Why LU state-space model?

- an alternative to the Kalman filtering
- inspired by "unknown-but-bounded" errors (deterministic)
- restriction both on states and parameters
- noise boundary estimation

$$\begin{aligned} x_t &= Ax_{t-1} + Bu_t + F + {}^x e_t \\ y_t &= Cx_t + Du_t + G + {}^y e_t \end{aligned}$$

where

$x_t, \ u_t, \ y_t$ mean state, input and output, respectively,

$\Theta = (A, \ B, \ C, \ D, \ F, \ G)$ are model parameters

${}^x e_t, \ {}^y e_t$ are state and output innovations, respectively.

The innovations have **uniform distributions**

$$\begin{aligned} f\left({}^x e_t\right) &= \mathcal{U}\left(0, {}^x r\right) \\ f\left({}^y e_t\right) &= \mathcal{U}\left(0, {}^y r\right) \end{aligned}$$

LINEAR UNIFORM STATE-SPACE MODEL

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JOINT PDF FOR THE STATE MODEL

Assumptions:
- natural conditions of control hold
- x_0, ${}^x r$, ${}^y r$, Θ are given

Then

$$f(d^{1:t}, x^{1:t} | x_0, {}^x r, {}^y r, \Theta) \propto \prod_{i=1}^{n} {}^x r_i^{-t} \prod_{j=1}^{m} {}^y r_j^{-t} \chi(\mathcal{S}),$$

The convex set $\mathcal{S}$ is given by inequalities

$$\begin{aligned} -{}^x r &\leq x_\tau - Ax_{\tau-1} - Bu_\tau - F &\leq {}^x r \\ -{}^y r &\leq y_\tau - Cx_\tau - Du_\tau - G &\leq {}^y r \end{aligned} \quad (1)$$

$\tau = 1, 2, \ldots, t,$

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$\tau = 1, 2, \ldots, t,$

If the inequalities (1) are linear in the unknowns and ${}^{x}r < 1$, ${}^{y}r < 1$, then

MAP estimation $\Leftrightarrow$ LP problem

The standard form of LP is used (Matlab function linprog)

$$\text{Find a vector } X \text{ such that } J \equiv \mathcal{C}'X \rightarrow \min$$
$$\text{while } \mathcal{A}X \leq \mathcal{B},\ \underline{X} \leq X \leq \overline{X}$$

where

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are known matrices derived from (1)

$\underline{X}$, $\overline{X}$ are known vectors defined by the prior pdf

Estimation of the state and the noise bounds

- knowns: parameters $\Theta = (A,\ B,\ C,\ D,\ F,\ G)$
- unknowns: the state $x^{1:t}$ and the noise bounds ${}^{x}r,\ {}^{y}r$
- $X = \left[(x^{1:t})',\ {}^{x}r',\ {}^{y}r'\right]'$

Estimation of the parameters and the noise bounds

- known: state trajectory $x^{1:t}$
- unknowns: parameters $\Theta = (A, B, C, D, F, G)$ and ${}^{x}r$, ${}^{y}r$
- $X \equiv [\mathrm{col}(A)', \mathrm{col}(B)', \mathrm{col}(C)', \mathrm{col}(D)', \mathrm{col}(F)', \mathrm{col}(G)', {}^{x}r', {}^{y}r']'$,
- $\mathrm{col}(A)$ transform the matrix A into the column vector

Joint parameter and state estimation

Swapping between the parameter and state estimation.

- the state $x^{1:t}$ is estimated with parameters Θ fixed at their last point estimates
- the resulting estimates of states, $\hat{x}^{1:t}$ are subsequently used to obtain new estimates of the parameters Θ

Linearization of the non-linear expressions at the newest point estimates.

- the expressions in (1) are approximated by the 1st order Taylor expansion $\Rightarrow$ linear inequalities
- the inequalities are transformed into the standard form of LP

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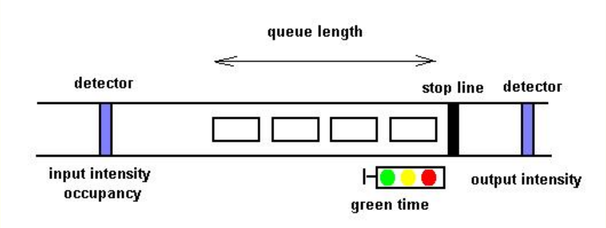
OFF-LINE $\longrightarrow$ ON-LINE MODE

- moving window

$$x^{1:t} \longrightarrow x^{(t-\partial):t}$$

- ∂ is memory length

Traffic data - basic characteristics



- input intensity I_t, output intensity Y_t [cars/time]
- occupancy O_t [%]
- green time z_t [%]
- saturated flow S [cars/time]
- sampling period T_p [time]
- queue length ξ_t [cars]
- queue indicator $\delta_t \in \{0,\ 1\}$

State equation

$$
\begin{aligned}
\xi_{t+1} &= \delta_t \xi_t - [\delta_t S + (1-\delta_t) I_t] z_t + I_t &+ {}^{x}e_{t;1} \\
O_{t+1} &= \kappa_t \xi_t + \beta_t O_t + \lambda_t &+ {}^{x}e_{t;2}
\end{aligned}
$$

Output equation

$$
\begin{aligned}
Y_{t+1} &= \Delta \xi_{t+1} + I_t &+ {}^{y}e_{t;1} \\
O_{t+1} &= O_{t+1} &+ {}^{y}e_{t;1}
\end{aligned}
$$

State equation

$$
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O_{t+1} &= O_{t+1} &&+ {}^y e_{t;1}
\end{aligned}
$$

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t z_t + F_t + {}^{x}e_t \\ y_{t+1} &= C_t x_t + D_t z_t + G_t + {}^{y}e_t \end{aligned}$$

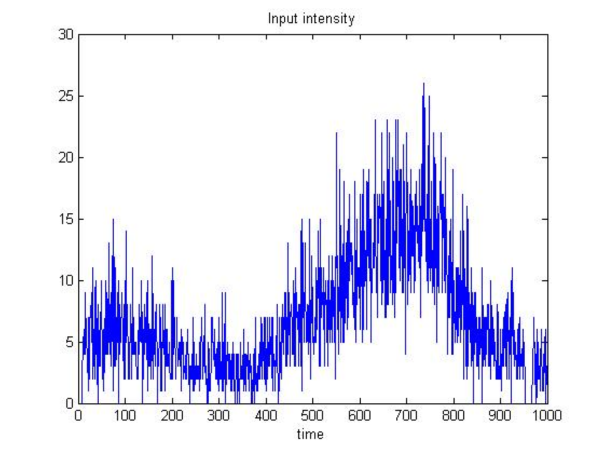
where

$$A_t = \begin{bmatrix} \delta_t & 0 \\ \kappa & \beta \end{bmatrix}, \quad B_t = \begin{bmatrix} -\delta_t S - (1-\delta_t) I_t \\ 0 \end{bmatrix}, \quad F_t = \begin{bmatrix} I_t \\ \lambda \end{bmatrix}$$

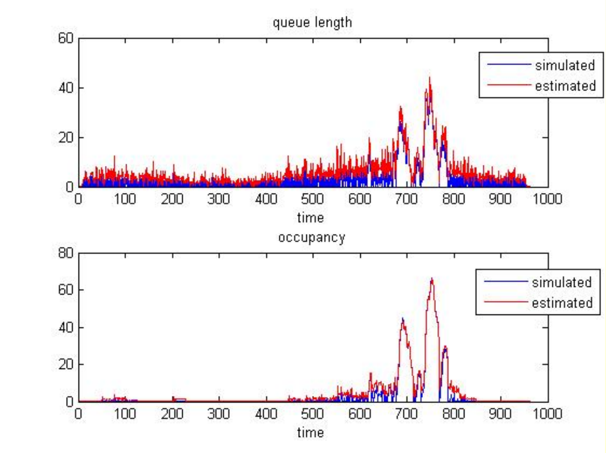
$$C_t = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad G_t = \begin{bmatrix} \hat{\xi}_t + I_t \\ 0 \end{bmatrix}$$

EXPERIMENTS - QUEUE LENGTH ESTIMATION

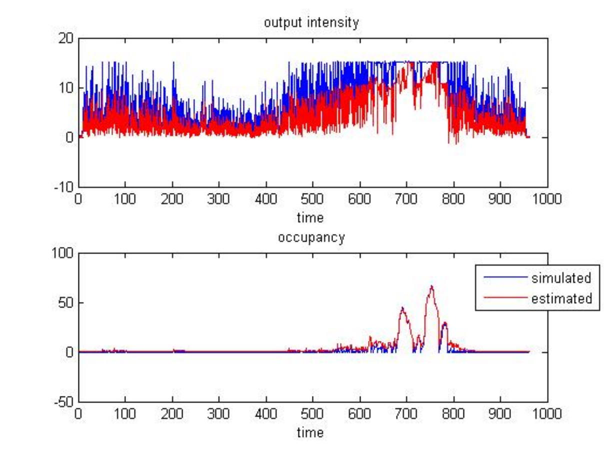
Given: Int, κ, β, λ, z, S
Unknown: Ynt, ξ



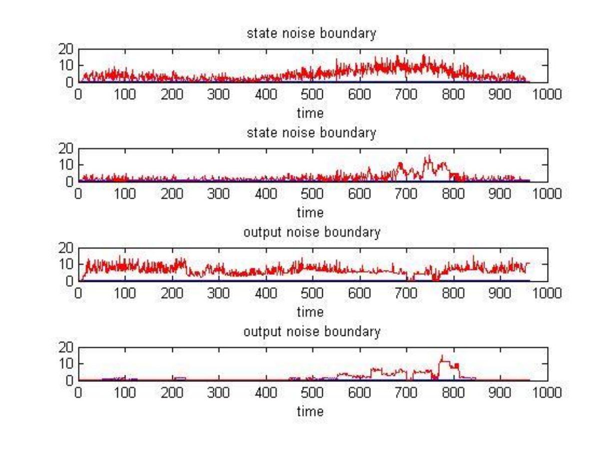
Queue length estimation



Output intensity prediction



Noise boundary estimation



Summary

- good results for single state/parameter estimation
- joint swapping based estimation depends on quality of initial estimates
- joint expansion based estimation gives reasonable results only for the output prediction

CONCLUSIONS

- exploitation in the cases when the models with unbounded support (KF) do not suite
- possibility of the noise-boundary estimation
- the approximation of the posterior pdf allows the recursive run
- restriction on the states decreases parameterization ambiguity

FUTURE PLANS

- choice of the optimal memory length
- search for more suitable expansion point
- extension to the non-uniform distributions with restricted support

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