Abstract

Goodness-of-fit testing is considered based on the statistics which are ϕ-divergences or ϕ-disparities between hypothetical and empirical distributions. Absolutely continuous distributions on $I\!R$ stand for the hypothetical distributions and density estimates based on spacings or histograms obtained from i.i.d. observations represent the empirical distributions. All these estimates can be obtained from quantiles of the standard empirical distribution functions. It is shown that the goodness-of-fit statistics considered in the previous literature are special cases of ϕ-divergence statistics. The main attention is paid to asymptotic properties of the ϕ-divergence and ϕ-disparity statistics based on spacings. Asymptotic equivalence is proved under various approaches to the definition of spacings which appeared in the previous literature. General law of large numbers and asymptotic normality theorem under local alternatives are proved from which one can obtain many previous asymptotic results as particular cases. Special attention is devoted to the asymptotic laws for the power divergence statistics of orders $\alpha \in (-1, \infty)$. Parameters of these laws are evaluated in a closed form and their continuity on the interval $(-1, \infty)$ is proved. These parameters are used to evaluate the local asymptotic power of the tests based on these statistics. This enables to extend previous results about asymptotic optimality of the statistics of power $\alpha = 2$ to the class of all statistics of the powers $\alpha \in (-1, \infty)$.

Key words:

Goodness-of-fit, Spacings, ϕ-divergences, ϕ-disparities, Power divergences, Asymptotic laws, Asymptotic normality, Asymptotic optimality.

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Goodness-of-fit tests based on observations quantized by hypothetical and empirical quantiles[1]

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1 Introduction and basic concepts

Goodness-of-fit tests decide about a hypothesis $\mathcal{H}_0 : F = F_0$ concerning an unknown distribution function $F(x)$, $x \in I\!R$ of independent observations $X_1, \ldots, X_n$. The decision is based on the order statistics

$$(Y_1, \ldots, Y_n) = (X_{n:1}, \ldots, X_{n:n}) \tag{1.1}$$

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which are sufficient functions of the observations. Another obvious sufficient statistic is the empirical distribution function

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(x \geq Y_i) = \frac{1}{n}\sum_{i=1}^{n} I(x \geq X_i), \quad x \in \mathbb{R} \tag{1.2}$$

where $I(\cdot)$ is the indicator function.

Intuitively one can expect that all goodness-of-fit test statistics will be measures of disparity between the distributions F_0 and F_n. For some statistics this is clear, e. g. the measure of disparity for the well known statistic of Kolmogorov (1941) and Smirnov (1944) is the Kolmogorov distance

$$K(F_0, F_n) = \sup_{x \in \mathbb{R}} |F_0(x) - F_n(x)|. \tag{1.3}$$

The first aim of the present paper is to show that the best known test statistics are measures of ϕ-divergence of Csiszár (1963) between F_0 and F_n, or measures of ϕ-disparities which are extensions of the ϕ-divergences introduced by Lindsay (1994) and more systematically studied by Menéndez et al (1998).

Let $\mathbf{\Phi}_0$ be the class of all continuous functions $\phi : (0, \infty) \mapsto \mathbb{R}$ which are strictly convex at 1 with $\phi(1) = 0$, and let us consider for every $\phi \in \mathbf{\Phi}_0$ the integral

$$D_\phi(F_0, F) = \int_{\mathbb{R}} \frac{\mathrm{d}F}{\mathrm{d}G}\, \phi\left(\frac{\mathrm{d}F_0/\mathrm{d}G}{\mathrm{d}F/\mathrm{d}G}\right) \mathrm{d}G, \quad G = \frac{F_0 + F}{2}, \tag{1.4}$$

where $\mathrm{d}F_0/\mathrm{d}G$, $\mathrm{d}F/\mathrm{d}G$ are the Radon–Nikodym densities of the distributions F_0, F with respect to the dominating distribution G, and where the conventions

$$\phi(0) = \lim_{t \downarrow 0} \phi(t), \quad 0\,\phi\left(\frac{s}{0}\right) = s \lim_{t \to \infty} \frac{\phi(t)}{t} \quad \text{for } s > 0 \text{ and } 0\,\phi\left(\frac{0}{0}\right) = 0 \tag{1.5}$$

are adopted behind the integral.

Let $\phi \in \mathbf{\Phi}_0$ be convex on the whole domain $(0, \infty)$. Then

$$\frac{\phi(t) - \phi(1)}{t - 1} = \frac{\phi(t)}{t - 1}$$

is increasing (nondecreasing) on each of the intervals $(0, 1)$ and $(1, \infty)$. This means that the limits $\phi(0)$ and $0\,\phi(1/0)$ assumed in (1.5) exist and also that the right-hand derivative $\phi'_+(1)$ exists, such that the difference $\phi(t) - \phi'_+(1)\,(t - 1)$ is nonnegative on $(0, \infty)$. Therefore $D_\phi(F_0, F)$ is well defined by (1.4) and (1.5) and called ϕ-divergence of F_0 and F, cf. Csiszár (1963).

Let us now consider $\phi \in \mathbf{\Phi}_0$ for which the limit $0\,\phi(1/0)$ of (1.5) and the right-hand derivative $\phi'_+(1)$ exist and the difference $\phi(t) - \phi'_+(1)\,(t - 1)$ is monotone on each of the intervals $(0, 1)$ and $(1, \infty)$. This implies that the limit $\phi(0)$ of (1.5) exists. Further, since

$\phi(t)$ is strictly convex at $t = 1$, this implies that $\phi(t) - \phi'_+(1)\,(t-1)$ is decreasing (nonincreasing) on $(0,1)$ and increasing (nondecreasing) on $(1,\infty)$. By assumption $\phi(1) = 0$ so that this means that $\phi(t) - \phi'_+(1)\,(t-1)$ is nonnegative on $(0,\infty)$. Consequently, $D_\phi(F_0, F)$ is well defined by (1.4) and (1.5) and called ϕ-disparity of F_0 and F, cf. Menéndez et al (1998).

If $\phi \in \mathbf{\Phi}_0$ is convex on $(0,\infty)$ then it satisfies the assumptions of the previous paragraph. Therefore the ϕ-divergences form a subclass in the class of ϕ-disparities. It is easy to verify (cf. Menéndez et al (1998)) that each ϕ-disparity $D_\phi(F_0, F)$ takes on values from the interval $[0,\ \phi(0) + 0\,\phi(1/0)]$ and the extremal values

$$D_\phi(F_0, F) = 0 \quad \text{or} \quad D_\phi(F_0, F) = \phi(0) + 0\,\phi\left(\frac{1}{0}\right)$$

are attained if and only if $F_0 = F$ or if F is supported by a subset $S \subset I\!\!R$ of zero F_0-probability, respectively. Further, if F_0 and F are absolutely continuous on $I\!\!R$ with densities f_0 and f (in symbols, $F_0 \sim f_0$ and $F \sim f$), then (1.4) reduces to

$$D_\phi(F_0, F) = \int_{I\!\!R} f\,\phi\left(\frac{f_0}{f}\right) \mathrm{d}x \tag{1.6}$$

where the conventions (1.5) are adopted behind the integral.

If F_0 is absolutely continuous then the support of the empirical distribution F_n has a.s. the zero F_0-probability, so that $D_\phi(F_0, F_n)$ is a.s. constant equal $\phi(0) + 0\,\phi(1/0)$. To overcome this problem we restrict the distributions F_0, F_n on the finite subfield of the Borel field generated by partitions

$$\mathcal{P} = \{(a_{j-1}, a_j] : 1 \le j \le k\}, \quad -\infty = a_0 < a_1 < \cdots < a_k = \infty \tag{1.7}$$

of $I\!\!R$. These partitions may depend on the sample size n in the sense that both the partition size k and the cutpoints $a_1, \ldots, a_{k-1}$ themselves depend on n, but this is not explicitly denoted in the paper. In this manner we obtain from (1.4) the ϕ-disparities or ϕ-divergences

$$D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = \sum_{j=1}^{k} p_{nj}\,\phi\left(\frac{p_{0j}}{p_{nj}}\right) \quad \text{(see (1.5))} \tag{1.8}$$

of the discrete distributions

$$\boldsymbol{p}_0 = (p_{0j} = F_0(a_j) - F_0(a_{j-1}) : 1 \le j \le k), \quad \boldsymbol{p}_n = (p_{nj} = F_n(a_j) - F_n(a_{j-1}) : 1 \le j \le k) \tag{1.9}$$

resulting from the original distributions F_0, F_n restricted on the partition (1.7). Since $\boldsymbol{p}_0$ a.s. dominates $\boldsymbol{p}_n$, the ϕ-disparities $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ cannot be a.s. constant, they discriminate $\boldsymbol{p}_n$ closer to $\boldsymbol{p}_0$ from those which are less close.

In this paper the attention is focused on the class of statistics

$$T_\phi = n\, D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n), \quad \phi \in \boldsymbol{\Phi} \tag{1.10}$$

where $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ is defined by (1.8) and $\boldsymbol{\Phi}$ is the class of continuous functions $\phi : (0,\infty) \mapsto \mathbb{R}$ with $\phi(t)$ monotone in the neighborhood of 0 and ∞ and $\phi(t)/t$ monotone in the neighborhood of ∞ which are twice continuously differentiable in a neighborhood of 1 with the second derivative $\phi''(1) > 0$ and $\phi(1) = 0$. Obviously, the limits $\phi(0)$ and $0\phi(s/0)$ considered in (1.5) exist and the sum (1.8) is well defined for all pairs $\boldsymbol{p}_0$, $\boldsymbol{p}_n$. If $\phi \in \boldsymbol{\Phi}$ is convex on $(0,\infty)$ then T_ϕ is a measure of ϕ-divergence of F_0 and F_n and if $\phi(t) - \phi'(1)\,(t-1)$ is monotone on $(0,1)$ and $(1,\infty)$ then it is a measure of ϕ-disparity of F_0 and F_n.

In Section 2 we recall the known fact that the classical goodness-of-fit statistics of Pearson (1900), Neyman and Pearson (1928), Neyman (1949) and Freeman–Tukey (1950) are ϕ-divergence measures from the class (1.10). We also show that the Anderson–Darling and Cramér–von Mises statistics are weighted averages of some ϕ-divergence statistics from the class (1.10). However, the main attention of Section 2 is payed to the goodness-of-fit statistics based on spacings. Various statistics of this type were introduced and studied by Greenwood (1946), Moran (1951), Darling (1953), Pyke (1965, 1972), Cressie (1976), Dudewicz and van der Meulen (1981), Hall (1984, 1986), Jammalamadaka et al (1989), Guttorp and Lockart (1989), van Es (1992), Shao and Hahn (1995), Ekström (1999), Misra and van der Meulen (2001), Morales et al (2003) and others cited there.

Spacings are obtained as components of the hypothetic distribution $\boldsymbol{p}_0$ defined in (1.9) when the cutpoints $a_1, \ldots, a_{k-1}$ of the partition (1.7) are selected from the order statistics $Y_1, \ldots, Y_n$. To present this idea in more detail, denote by

$$F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}, \quad \alpha \in (0,1)$$

the quantile function of an arbitrary distribution F and replace in the definition of cutpoints

$$a_j = F_0^{-1}(j/k), \quad 1 \leq j \leq k-1 \tag{1.11}$$

the hypothetic quantiles by similar empirical quantiles, i. e. let the cutpoints be random, defined by formula

$$a_j = F_n^{-1}(j/k), \quad 1 \leq j \leq k-1. \tag{1.12}$$

If k depends on the sample size so that $n = mk$ for a fixed integer $m \geq 1$, then we get the order statistics cutpoints $a_j = Y_{mj}$ for $1 \leq j \leq k-1$. If F_0 is continuous then the hypothetic quantiles (1.11) lead to the uniform hypothetic distribution (1.9),

$$\boldsymbol{p}_0 = \left(p_{0j} = \frac{1}{k} : 1 \leq j \leq k\right).$$

The empirical quantiles (1.12) lead to the uniform empirical distribution (1.9),

$$\boldsymbol{p}_n = \left(p_{nj} = \frac{1}{k} : 1 \le j \le k \right)$$

and to the random theoretical distribution $\boldsymbol{p}_0$ with probabilities

$$p_{0j} = \left\langle \begin{array}{ll} F_0(Y_{mj}) - F_0(Y_{m(j-1)}) & \text{for} \quad 1 \le j \le k-1 \\ F_0(Y_{n+1}) - F_0(Y_{m(k-1)}) & \text{for} \quad j = k \end{array} \right.$$

with the dummy observations $Y_0 = -\infty$ and $Y_{n+1} = \infty$.

Let the observation space be reduced to the interval $[0,1] \subset I\!R$ and $F_0(x) = x$ for $x \in [0,1]$. Then the last formula yields the nonoverlapping spacings

$$p_{0j} = \left\langle \begin{array}{ll} Y_{mj} - Y_{m(j-1)} & \text{for} \quad 1 \le j \le k-1 \\ Y_{n+1} - Y_{m(k-1)} & \text{for} \quad j = k \end{array} \right.$$

with the dummy observations

$$Y_0 = 0 \quad \text{and} \quad Y_{n+1} = 1 \tag{1.13}$$

studied e. g. by Del Pino (1979) and Jammalamadaka et al (1989). We are interested in the simple spacings where $m = 1$ and $k = n$. Then the distributions (1.9) take on the form

$$\boldsymbol{p}_0 = \left(p_{0j} = \left\langle \begin{array}{ll} Y_j - Y_{j-1} & \text{for} \quad 1 \le j \le n-1 \\ Y_{n+1} - Y_{n-1} & \text{for} \quad j = n \end{array} \right. \right), \quad \boldsymbol{p}_n = \left(p_{nj} = \frac{1}{n} : 1 \le j \le n \right). \tag{1.14}$$

Using the formula (1.8) we obtain from here that the statistics (1.10) take on the form

$$\begin{aligned} T_\phi &= S_\phi - \phi(n(Y_{n+1} - Y_n)) - \phi(n(Y_n - Y_{n-1})) + \phi(n(Y_{n+1} - Y_{n-1})) \\ \text{for} \quad S_\phi &= \sum_{j=1}^{n+1} \phi(n(Y_j - Y_{j-1})), \quad \phi \in \boldsymbol{\Phi} \end{aligned} \tag{1.15}$$

where Y_0 and Y_{n+1} are the same as in (1.13). This prescribes the exact form of the statistics using the information contained in the spacings $Y_j - Y_{j-1}$, $1 \le j \le n+1$, and based on the ϕ-divergence or ϕ-disparity of the hypothetic and empirical distributions F_0, F_n.

In Section 2 we list the statistics proposed for simple spacings in the above mentioned literature. All of them differ from T_ϕ of (1.15). However, in Section 3 we prove that they share all statistically relevant asymptotic properties with T_ϕ of (1.15) so that the differences are only numerical and not statistically principal. In Section 4 we evaluate the asymptotic means and variances of the statistics under consideration when $\phi \in \boldsymbol{\Phi}$ varies continuously in a real valued parameter and study their continuity in this parameter. This section parallels in some sense the effort of Read and Cressie summarized in their monograph of (1988). They have shown that various goodness-of-fit statistics based on the deterministic partitions (1.7) of the type (1.11) are just special cases of certain statistics T_{ϕ_α} defined by (1.10) for ϕ_α-divergences $D_{\phi_\alpha}(\boldsymbol{p}_0, \boldsymbol{p}_n)$ specified by convex functions ϕ_α continuously depending on the parameter $\alpha \in I\!R$. They proved that the most important properties of these statistics are shared for all $\alpha \in I\!R$, or at least for all α from large intervals on $I\!R$, so that the theory of a large class of goodness-of-fit statistics can be unified and simplified by increasing the level of mathematical abstraction. We show in Section 4 that the spacings-based goodness-of-fit statistics can similarly be unified and their theory simplified by treating the whole class T_{ϕ_α}, $\alpha \in I\!R$, obtained from (1.10) under the empirical quantile partitions (1.12).

2 ϕ-disparities and test statistics

In this section we study some concrete statistics T_ϕ from the class (1.10). We show that all common goodness-of-fit statistics are measures of disparity or divergence between the restrictions $\boldsymbol{p}_0$, $\boldsymbol{p}_n$ of F_0, F_n belonging to this class.

Our first aim are the statistics defined by the partitions (1.7) with cutpoints a_j defined by a deterministic rule, e. g. by (1.11). We show that the most common statistics are in this case measures of ϕ-divergence defined in accordance with the formula (1.8) for convex functions $\phi \in \boldsymbol{\Phi}$. Obviously, the choice $\phi(t) = (t-1)^2/t$ leads to the Pearson (1900) statistic

$$T = n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0) = n\sum_{j=1}^k \frac{(p_{nj} - p_{0j})^2}{p_{0j}} = \sum_{j=1}^k \frac{(Z_j - np_{0j})^2}{np_{0j}} \tag{2.1}$$

where $(Z_j : 1 \le j \le k) = (np_{nj} : 1 \le j \le k)$ is multinomially distributed with parameters n and

$$\boldsymbol{p} = (p_j = F(a_j) - F(a_{j-1}) : 1 \le j \le k) \tag{2.2}$$

being the restriction of the true distribution F on the partition (1.7). Similarly, $\phi(t) = -2\ln t$ and $\phi(t) = 2t\ln t$ lead to the log-likelihood statistic

$$T = 2n\, I(\boldsymbol{p}_n, \boldsymbol{p}_0) = 2n\sum_{j=1}^k p_{nj} \ln\frac{p_{nj}}{p_{0j}} = 2\sum_{j=1}^k Z_j \ln\frac{Z_j}{np_{0j}} \tag{2.3}$$

and the reversed log-likelihood statistic

$$T = 2n\, I(\boldsymbol{p}_0, \boldsymbol{p}_n) = 2n\sum_{j=1}^k p_{0j} \ln\frac{p_{0j}}{p_{nj}} = 2\sum_{j=1}^k np_{0j} \ln\frac{np_{0j}}{Z_j} \tag{2.4}$$

of Neyman and Pearson (1928), $\phi(t) = (t-1)^2$ leads to the Neyman (1948) statistic

$$T = n\,\chi^2(\boldsymbol{p}_0, \boldsymbol{p}_n) = n\sum_{j=1}^k \frac{(p_{nj} - p_{0j})^2}{p_{nj}} = \sum_{j=1}^k \frac{(Z_j - np_{0j})^2}{Z_j} \tag{2.5}$$

and $\phi(t) = 4(1-\sqrt{t})$ leads to the Freeman–Tukey (1950) statistic

$$T = 8n\, H(\boldsymbol{p}_0, \boldsymbol{p}_n) = 8n\left(1 - \sum_{j=1}^{k} \sqrt{p_{nj}p_{0j}}\right) = 4\sum_{j=1}^{k}\left(\sqrt{Z_j} - \sqrt{np_{0j}}\right)^2. \tag{2.6}$$

In these formulas we used the symbols $\chi^2(\boldsymbol{p}_0, \boldsymbol{p}_n)$, $I(\boldsymbol{p}_0, \boldsymbol{p}_n)$ and $H(\boldsymbol{p}_0, \boldsymbol{p}_n)$ for the χ^2-divergence, I-divergence and Hellinger divergence defined by (1.8) for $\phi(t) = (t-1)^2$, $\phi(t) = t\ln t$ and $\phi(t) = 1-\sqrt{t}$, respectively. These best known ϕ-divergences will be often used in the sequel. For the terminology and more details about the statistics (2.1)–(2.6) we refer to Read and Cressie (1988).

Notice that the Kolmogorov distance $K(F_0, F_n)$ is the maximal ϕ-divergence of restrictions $\boldsymbol{p}_{0x} = (F_0(x),\, 1 - F_0(x))$ and $\boldsymbol{p}_{nx} = (F_n(x),\, 1 - F_n(x))$ of the original distributions F_0 and F_n on the class of binary partitions

$$\mathcal{P}_x = \{(-\infty, x], (x, \infty)\}, \quad x \in I\!R$$

of $I\!R$ for the convex function $\phi(t) = |t-1|/2$ not differentiable at $t = 1$. For smooth functions $\phi(t)$ from $\boldsymbol{\Phi}$ it is convenient to replace $\sup_{x\in I\!R} D_\phi(\boldsymbol{p}_{0x}, \boldsymbol{p}_{nx})$ by the average values

$$D_\phi(F_0, F_n|W) = \int_{I\!R} W(x)\, D_\phi(\boldsymbol{p}_{0x}, \boldsymbol{p}_{nx})\, \mathrm{d}F_0(x) \tag{2.7}$$

taken with respect to continuous weights $W : I\!R \to [0, \infty)$. For example, for the convex function $\phi(t) = (t-1)^2/t$ leading to the Pearson statistic (2.1) we obtain the average divergence

$$D(F_0, F_n|W) = \int_{I\!R} W(x)\, \frac{(F_0(x) - F_n(x))^2}{F_0(x)\,(1 - F_0(x))}\, \mathrm{d}F_0(x). \tag{2.8}$$

The statistic $T_W = n\, D(F_0, F_n|W)$ reduces for the weight $W(x) \equiv 1$ or $W(x) = F_0(x)\,(1 - F_0(x))$ to the Anderson–Darling or Cramér–von Mises goodness-of-fit statistic respectively, see Durbin (1973), Anderson and Darling (1954), von Mises (1947) and also pp. 58–64 in Serfling (1980). Basic results about the general class of statistics $T_{\phi,W} = n\, D_\phi(F_0, F_n|W)$ were overviewed in Darling (1957).

Let us now turn to the class of statistics, T_ϕ defined by (1.15). We show that the statistics based in the literature on simple spacings can be viewed as measures of ϕ-disparity between F_0 and F_n from this class. By this we mean that they are in some sense equivalent to the statistics T_ϕ defined by (1.15) for $\phi \in \boldsymbol{\Phi}$ with monotone differences $\phi(t) - \phi'(1)\,(t-1)$ on the intervals $(0,1)$ and $(1,\infty)$. For the best known statistics based on spacings the corresponding functions $\phi \in \boldsymbol{\Phi}$ are convex on $(0,\infty)$, i.e. these statistics are measures of ϕ-divergence between F_0 and F_n.

In accordance with the literature dealing with testing of hypotheses based on spacings, in the rest of this section, and in the rest of paper, we suppose that the distribution F is concentrated on the interval $(0,1]$ and that F_0 is uniform on this interval, i.e. $F_0(x) = x$, $x \in [0,1]$. Then all interval partitions of $\mathbb{R}$ under consideration can be reduced to the partitions of $(0,1]$, i.e. we put $a_0 = 0$ and $a_k = 1$ in (1.7).

We start with the simplest and best known case where $m = 1$ and $k = n$ in (1.12) leading to the cutpoints $a_j = Y_j$, $1 \leq j \leq n-1$ in the interval $(0,1]$. By (1.14), the components of the null distribution $\boldsymbol{p}_0$ are in this case

$$p_{0j} = \begin{cases} Y_1 & j = 1 \\ Y_j - Y_{j-1} \quad \text{for} & 2 \leq j \leq n-1 \\ 1 - Y_{n-1} & j = n \end{cases} \tag{2.9}$$

and the empirical distribution $\boldsymbol{p}_n = (1/n, \ldots, 1/n)$ is uniform. From (1.8) and (1.10) we obtain the class of statistics

$$T_\phi = n\, D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = \sum_{j=1}^{n} \phi(n\, p_{0j}), \quad \phi \in \boldsymbol{\Phi} \tag{2.10}$$

as considered already in (1.15). These statistics are measures of ϕ-divergence or ϕ-disparity between the distributions F_0 and F_n if $\phi(t)$ is convex on $(0,\infty)$ or the difference $\phi(t) - \phi'(t)\,(t-1)$ is monotone on $(0,1)$ and $(1,\infty)$, respectively.

The authors dealing with the statistics based on differences between order statistics (spacings) introduced a number of modifications of the statistics (2.10). These modifications depend on various possibilities to represent the tail probabilities $p_{01} = Y_1$ and $p_{0n} = 1 - Y_{n-1}$ as spacings. One possibility is to introduce artificial observations $Y_0 = 0$ and $Y_{n+1} = 1$ which was already done in (1.13) and which leads to the spacings

$$p_{01} = Y_1 - Y_0, \quad \tilde{p}_{0n} = Y_n - Y_{n-1}, \quad \tilde{p}_{0,n+1} = Y_{n+1} - Y_n. \tag{2.11}$$

Some authors adopted this approach and studied the statistics

$$S_\phi = \sum_{j=1}^{n-1} \phi(n\, p_{0j}) + \phi(n\, \tilde{p}_{0n}) + \phi(n\, \tilde{p}_{0,n+1}) \tag{2.12}$$

previously introduced in (1.15) (e.g. Jammalamadaka et al (1986, 1989)). Some authors neglected the tail probabilities $p_{01} = Y_1$ and $\tilde{p}_{0,n+1} = 1 - Y_n$ and studied the statistics

$$\tilde{S}_\phi = S_\phi - \phi(np_{01}) - \phi(n, \tilde{p}_{0,n+1}) = \sum_{j=2}^{n} \phi(n(Y_j - Y_{j-1})) \tag{2.13}$$

(e.g. Hall (1984)). Many authors studied the following modification of S_ϕ

$$S_\phi^+ = \sum_{j=1}^{n+1} \phi((n+1)\,(Y_j - Y_{j-1})) \tag{2.14}$$

(see Ekström (1999), Misra and van der Meulen (2001) and others cited by them). Another possibility is to interpret the observation space $(0,1]$ as a circle of unit circumference and to use $a_j = Y_j$, $1 \le j \le n-1$ considered above and also $a_n = Y_n$ as cutpoints of an interval partition $\{\tilde{A}_j : 1 \le j \le n\}$ on this circle. This will join the intervals $A_1 = (0, Y_1]$ and $A_{n+1} = (Y_n, Y_{n+1}]$ of the interval partition $\{A_j = (Y_{j-1}, Y_j] : 1 \le j \le n+1\}$ of $(0,1]$ into one interval $\tilde{A}_1 = A_1 \cup A_{n+1}$ on the circle and, consequently, merge the probabilities $p_{01} = Y_1 - Y_0 = Y_1$ and $\tilde{p}_{0,n+1} = Y_{n+1} - Y_n = 1 - Y_n$ into

$$\tilde{p}_{01} = p_{01} + \tilde{p}_{0,n+1} = 1 + Y_1 - Y_n.$$

This leads to the new theoretical distribution

$$\tilde{\boldsymbol{p}}_0 = (\tilde{p}_{01}, p_{02}, \dots, p_{0,n-1}, \tilde{p}_{0n})$$

with $p_{02}, \dots, p_{0,n-1}$ and $\tilde{p}_{0n}$ defined by (2.9), (2.11) and to the same uniform distribution $\boldsymbol{p}_n$ as before. With this approach our statistics T_ϕ of (2.10) are replaced by

$$\tilde{T}_\phi = n\, D_\phi(\tilde{\boldsymbol{p}}_0, \boldsymbol{p}_n) = \sum_{j=2}^{n-1} \phi(n\, p_{0j}) + \phi(n\, \tilde{p}_{01}) + \phi(n\, \tilde{p}_{0n}), \quad \phi \in \boldsymbol{\Phi}. \tag{2.15}$$

Some authors (e. g. Hall (1986)) used the statistics

$$\tilde{T}_\phi^+ = \sum_{j=2}^{n-1} \phi((n+1)\, p_{0j}) + \phi((n+1)\, \tilde{p}_{01}) + \phi((n+1)\, \tilde{p}_{0n}). \tag{2.16}$$

It is to be noted that Ekström (1999) and most authors cited by him studied the statistic S_ϕ^+ only with the convex function $\phi(t) = -\ln t$ belonging to $\boldsymbol{\Phi}$ while Misra and van der Meulen (2001) studied $\phi(t) = t \ln t$. On the other hand, Hall (1986), Jammalamadaka et al (1989), Guttorp and Lockart (1989) and others studied the statistics S_ϕ, $\tilde{S}_\phi$ or S_ϕ^+ for ϕ from a wider class $\tilde{\boldsymbol{\Phi}} = \{c_1\phi + c_2 : c_1,\, c_2 \in I\!R,\, \phi \in \boldsymbol{\Phi}\}$ than $\boldsymbol{\Phi}$. However, if $\tilde{\phi} \in \tilde{\boldsymbol{\Phi}}$ then for every statistic $U_{\tilde{\phi}}$ from the class $\{S_{\tilde{\phi}}, \tilde{S}_{\tilde{\phi}}, S_{\tilde{\phi}}^+\}$ there exist $c_1,\, c_2 \in I\!R$ and a function $\phi \in \boldsymbol{\Phi}$ such that

$$U_{\tilde{\phi}} = c_1 U_\phi + c_2 \quad \text{for some } U_\phi \in \{S_\phi, \tilde{S}_\phi, S_\phi^+\}.$$

This means that the functions considered by these authors can be restricted without loss of generality to those from $\boldsymbol{\Phi}$. Further, the assumption $\phi''(1) > 0$ for $\phi \in \boldsymbol{\Phi}$ implies that ϕ is strictly convex in a neighborhood of 1. Consequently, $\phi(t) - \phi'(1)\,(t-1)$ is decreasing on some interval $(a,1) \subset (0,1)$ and increasing on $(1,b) \subset (0,\infty)$. Since there is no visible reason for considering $\phi(t)$ oscillating on $(0,a)$ or (b,∞) if these intervals are nonvoid, we can assume without loss of generality that the functions ϕ proposed by the mentioned authors define ϕ-disparities of probability distributions. Combining this result with the fact that the differences $T_\phi - U_\phi$ and $\tilde{T}_\phi - U_\phi$ are for all $U_\phi \in \{S_\phi, \tilde{S}_\phi, S_\phi^+, \tilde{T}_\phi^+\}$

statistically negligible, when $n \to \infty$ (see Section 3 below), we can conclude that the goodness-of-fit statistics based on simple spacings are in fact measures of ϕ-disparity or of ϕ-divergence between the hypothetical and empirical distributions F_0 and F_n. Similar conclusion can be obtained also for the statistics based on m-spacings for fixed $m > 1$, and also for $m = m_n \to \infty$ as $n \to \infty$ (these cases are not considered in the present paper; the subcase $m_n/n \to \infty$ has been analyzed recently by Morales et al (2003)).

3 General asymptotic results

In this section we study the finite set of statistics

$$\{T_\phi, \tilde{T}_\phi, \tilde{T}_\phi^+, S_\phi\} \tag{3.1}$$

for all ϕ from the set $\mathbf{\Phi}$ defined in (1.10). The statistics of (3.1) were defined in Section 2. Here we extend the asymptotic results proved previously for one of the statistics (3.1) to all the statistics of (3.1). This extension is achieved at the price of a restriction on the set $\mathbf{\Phi}$, namely we consider the subsets $\mathbf{\Phi}_2 \subset \mathbf{\Phi}_1 \subset \mathbf{\Phi}$ defined by the condition that there exist functions ξ, $\eta, \zeta : (0, \infty) \mapsto I\!R$ such that every $\phi \in \mathbf{\Phi}_1$ satisfies for all s, $t \in (0, \infty)$ the functional equation

$$\phi(st) = \xi(s)\,\phi(t) + \zeta(t)\,\phi(s) + \eta(s)\,(t-1) \tag{3.2}$$

and every $\phi \in \mathbf{\Phi}_2$ satisfies the functional equation

$$\phi(st) = \xi(s)\,\phi(t) + \phi(s) + \eta(s)\,(t-1). \tag{3.3}$$

Lemma 3.1. The functions ξ, ζ and η are continuous on $(0, \infty)$ and satisfy the relations

$$\xi(1) = \zeta(1) = 1 \quad \text{and} \quad \eta(1) = 0. \tag{3.4}$$

Proof. The continuity of ξ and η from (3.3) can be obtained by putting $s = 2$ and $t = 2$ and $t = 3$ in (3.2). If we put $s = 1$ in (3.2) or (3.3) and use the assumption $\phi(1) = 0$ then we obtain that for all $t \in (0, \infty)$

$$(\xi(1) - 1)\,\phi(t) + \eta(1)\,(t-1) = 0.$$

This contradicts the assumption $\phi''(1) > 0$ unless $\xi(1) = 1$ which implies also $\eta(1) = 0$. By putting $t = 1$ in (3.2) we find that $\zeta(1) = 1$. □

Lemma 3.2. Every $\phi \in \mathbf{\Phi}_1$ is differentiable on $(0, \infty)$, the corresponding functions ξ and η are differentiable at 1, and for every $t > 0$

$$\phi'(t) = \xi'(1)\,\frac{\phi(t)}{t} + \phi'(1)\,\frac{\zeta(t)}{t} + \eta'(1)\,\frac{t-1}{t}. \tag{3.5}$$

Proof. Putting $s = 1 + \varepsilon$ and

$$\xi^*(\varepsilon) = \frac{\xi(1+\varepsilon) - \xi(1)}{\varepsilon}, \quad \eta^*(\varepsilon) = \frac{\eta(1+\varepsilon) - \eta(1)}{\varepsilon}$$

we obtain from (3.2) for every $t > 0$ and ε close to 0

$$t\,\frac{\phi(t+\varepsilon t) - \phi(t)}{\varepsilon t} = \xi^*(\varepsilon)\,\phi(t) + \frac{\phi(1+\varepsilon) - \phi(1)}{\varepsilon}\,\zeta(t) + \eta^*(\varepsilon)\,(t-1). \tag{3.6}$$

Since ϕ is differentiable in a neighborhood of 1, for t close to 1

$$\xi^*(\varepsilon)\,\phi(t) + \eta^*(\varepsilon)\,(t-1) = t\,\phi'(t) - \phi'(1)\,\zeta(t) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

By assumptions concerning $\mathbf{\Phi}$, $\phi(t)$ is not linear in a neighborhood of $t = 1$. Therefore the last relation implies that the limits of $\xi^*(\varepsilon)$ and $\eta^*(\varepsilon)$ for $\varepsilon \to 0$ exist, i. e.,

$$\xi^*(\varepsilon) = \xi'(1) + o(\varepsilon) \quad \text{and} \quad \eta^*(\varepsilon) = \eta'(1) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Now (3.5) for all $t > 0$ follows from (3.6). □

Example 3.1. The function $\phi(t) = (1-t)/t$, $t > 0$, belongs to $\mathbf{\Phi}$ and satisfies (3.3) for $\xi(t) = 1/t$ and $\eta(t) \equiv 0$. Therefore it belongs to $\mathbf{\Phi}_2 \subset \mathbf{\Phi}$. The function $\phi(t) = (1-t)^2/t$, $t > 0$, belongs to $\mathbf{\Phi}$ too and satisfies (3.3) for the same $\xi(t)$ as above and $\eta(t) = t - 1/t$. Therefore it belongs to $\mathbf{\Phi}_2$. The class of functions defined on $(0, \infty)$ by

$$\phi_\alpha(t) = \frac{t^\alpha \ln t}{(2\alpha - 1)}, \quad \alpha \in I\!R - \{\tfrac{1}{2}\}$$

belong to $\mathbf{\Phi}$ and satisfy (3.2) for $\xi(t) = \zeta(t) = t^\alpha$ and $\eta(t) \equiv 0$. Therefore

$$\{\phi_\alpha : \alpha \in I\!R - \{\tfrac{1}{2}\}\} \subset \mathbf{\Phi}_1$$

and $\phi_0 \in \mathbf{\Phi}_2$. But ϕ_1 satisfies also (3.3) for $\xi(t) = t$ and $\eta(t) = t \ln t$. Therefore ϕ_1 belongs to $\mathbf{\Phi}_2$.

In the theorems that follow the observations are assumed to be distributed on $(0, 1]$ in two possible ways:

(i) under a fixed alternative,

(ii) under local alternatives.

The case (i) means that the observations are distributed by a fixed distribution function $F(x)$ with a density $f(x)$ positive if and only if $x \in [0, 1]$ and continuous on $[0, 1]$. The case (ii) means that the observations from samples of sizes $n = 1, 2, \ldots$ are distributed by distribution functions

$$F^{(n)}(x) = F_0(x) + \frac{L_n(x)}{\sqrt[4]{n}} = x + \frac{L_n(x)}{\sqrt[4]{n}} \tag{3.7}$$

on $[0,1]$ where $L_n : I\!R \mapsto I\!R$ are continuously differentiable functions with $L_n(0) = L_n(1) = 0$ with the derivatives $\ell_n(x) = L_n'(x)$ tending on $[0,1]$ to a continuously differentiable function $\ell : I\!R \mapsto I\!R$ uniformly in the sense

$$\sup_{0 \le x \le 1} |\ell_n(x) - \ell(x)| = o(1) \quad \text{as } n \to \infty. \tag{3.8}$$

The two possibilities (i) and (ii) are not mutually exclusive: their conjunction is "under the hypothesis" where $F(x) = F_0(x)$, $f(x) = f_0(x) = I_{[0,1]}(x)$ and $L_n(x) \equiv 0$ on $I\!R$ for all n. This means that the asymptotic results obtained under local alternatives for $\ell(x)$ of (3.8) equal identically 0 must coincide with the results obtained under the fixed alternative for $F(x) = F_0(x)$.

The theorems below demonstrate that if $\phi \in \mathbf{\Phi}_2$ defines a ϕ-divergence or ϕ-disparity then the statistics S_ϕ, $\tilde{S}_\phi$, S_ϕ^+ and T_ϕ^+ share the most important statistical properties with the ϕ-divergence or ϕ-disparity statistics T_ϕ and $\tilde{T}_\phi$. In other words, they provide a key argument for the thesis of the present paper formulated in Section 2, that the spacings-based goodness-of-fit statistics considered in the previous literature are measures of ϕ-divergence or ϕ-disparity between the hypothetic and empirical distributions F_0 and F_n. But independently of this purpose, these theorems present the asymptotic theory for the whole set of statistics (3.1) and clarify that the small modifications distinguishing these statistics from one another are asymptotically negligible. The restriction to the functions from $\mathbf{\Phi}_2$ or even $\mathbf{\Phi}_1$ is not essential – it only simplifies the proof of the next theorem.

Theorem 3.1. Consider the observations under fixed or local alternatives and denote by U_ϕ any statistic from the class $\{T_\phi, S_\phi, \tilde{T}_\phi\}$ defined in (2.10)–(2.15). For all $\phi \in \mathbf{\Phi}_1$

$$U_\phi - \tilde{S}_\phi = O_p(1) \quad \text{as } n \to \infty \tag{3.9}$$

and for all $\phi \in \mathbf{\Phi}_2$

$$S_\phi^+ - S_\phi = \varepsilon_n \, S_\phi + \delta_n \quad \text{and} \quad \tilde{T}_\phi^+ - \tilde{T}_\phi = \varepsilon_n \, \tilde{T}_\phi + \delta_n \tag{3.10}$$

where S_ϕ^+ and $\tilde{T}_\phi^+$ are defined by (2.14) and (2.16), $\varepsilon_n = o(1)$ and $\delta_n = \phi'(1) + o(1)$ as $n \to \infty$.

Proof. We shall consider the fixed alternative $F(x)$ with a continuous density $f(x) > 0$ for $0 \le x \le 1$. For the local alternatives the argument is similar. By inspecting the definitions of T_ϕ, $\tilde{T}_\phi$ and $\tilde{S}_\phi$ we see that for (3.9) it suffices to prove that for $n \to \infty$

$$\phi(np_{01}) = O_p(1) \quad \text{and} \quad \phi(n(p_{01} + p_{02})) = O_p(1). \tag{3.11}$$

It is known (see e. g. page 208 in Hall (1986)) that $p_{01} = F^{-1}(Z_1/W_{n+1})$ and $p_{01} + p_{02} = F^{-1}((Z_1+Z_2)/W_{n+1})$ where $Z_1, \dots, Z_{n+1}$ are independent standard exponential variables and $W_{n+1} = Z_1 + \cdots + Z_{n+1}$ so that, for $n \to \infty$,

$$\frac{W_{n+1}}{n} \xrightarrow{p} 1 \quad \text{and} \quad V_n = \frac{Z_1}{W_{n+1}} \xrightarrow{p} 0.$$

Setting

$$R_n = \frac{F^{-1}(V_n)}{V_n} = \frac{F^{-1}(V_n) - F^{-1}(0)}{V_n}$$

and using the mean value theorem and the assumed continuity of f in the neighborhood of 0, we find that

$$R_n \xrightarrow{p} \frac{1}{f(0)} \quad \text{as } n \to \infty$$

where, by assumptions about f, $0 < f(0) < \infty$. Thus

$$np_{01} = \frac{n}{W_{n+1}} Z_1 R_n$$

and, by applying (3.2),

$$\phi(np_{01}) = \xi\left(\frac{n}{W_{n+1}}\right) \phi(Z_1 R_n) + \zeta(Z_1 R_n)\, \phi\left(\frac{n}{W_{n+1}}\right) + \eta\left(\frac{n}{W_{n+1}}\right) (Z_1 R_n - 1).$$

Since $Z_1 R_n = O_p(1)$ as $n \to \infty$, we obtain from Lemma 3.1

$$\begin{aligned}
\phi(np_{01}) &= \left[\xi\left(\frac{n}{W_{n+1}}\right) + \phi\left(\frac{n}{W_{n+1}}\right) + \eta\left(\frac{n}{W_{n+1}}\right)\right] O_p(1) \\
&= [\xi(1) + \phi(1) + \eta(1) + o_p(1)]\, O_p(1) \\
&= O_p(1) \quad \text{(cf. (3.4))}.
\end{aligned}$$

Replacing $V_n = Z_1/W_{n+1}$, by $V_n = (Z_1 + Z_2)/W_{n+1}$ and using the fact that

$$(Z_1 + Z_2)\, R_n = (Z_1 + Z_2)\, \frac{F^{-1}(V_n) - F^{-1}(0)}{V_n} = O_p(1)$$

we obtain the second relation of (3.11). Now we prove (3.10). From (3.3) we get for any $p > 0$

$$\phi((n+1)\,p) = \xi\left(\frac{n+1}{n}\right) \phi(np) + \phi\left(\frac{n+1}{n}\right) + \eta\left(\frac{n+1}{n}\right) (np - 1)$$

so that

$$\phi((n+1)\,p) - \phi(np) = \varepsilon_n \phi(np) + \phi\left(\frac{n+1}{n}\right) + \eta\left(\frac{n+1}{n}\right)(np-1) \tag{3.12}$$

where $\varepsilon_n = \xi((n+1)/n) - 1 = o(1)$ as $n \to \infty$ by Lemma 3.1. Replacing p by p_{0j} figuring in the definitions of S_ϕ and S_ϕ^+ and summing over $1 \le j \le n+1$, we get the equality

$$S_\phi^+ - S_\phi = \varepsilon_n\, S_\phi + \delta_n$$

for

$$\begin{aligned} \delta_n &= (n+1)\,\phi\left(\frac{n+1}{n}\right) - \eta\left(\frac{n+1}{n}\right) \\ &= \frac{n+1}{n}\,\frac{\phi\left(1+\frac{1}{n}\right) - \phi(1)}{\frac{1}{n}} - \eta\left(\frac{n+1}{n}\right). \end{aligned}$$

By Lemma 3.1,

$$\delta_n = \phi'(1) + o(1) \quad \text{as } n \to \infty.$$

This completes the proof of the first relation in (3.10). Proof of the second relation is the same, we just replace p in (3.12) by the probabilities p_{0j} figuring in the definition of $\tilde{T}_\phi$. □

For every continuous function $\psi : (0,\infty) \mapsto \mathbb{R}$ we define the condition

$$\lim_{t\to\infty} t^{-\alpha}|\psi(t)| = \lim_{t\downarrow 0} t^{\beta}|\psi(t)| = 0 \quad \text{for some } \alpha \ge 0 \text{ and } \beta < 1 \tag{3.13}$$

and the integral

$$\langle \psi \rangle = \int_0^\infty \psi(t)\, e^{-t}\, \mathrm{d}t. \tag{3.14}$$

Obviously, if (3.13) holds then $\langle \psi \rangle$ exists and is finite.

Let $\phi \in \mathbf{\Phi}_1$ satisfy (3.13) and let

$$\xi = \xi_\phi, \quad \zeta = \zeta_\phi \quad \text{and} \quad \eta = \eta_\phi \tag{3.15}$$

be the corresponding functions satisfying the functional equation (3.2). Then all functions

$$\psi(t) = \phi(ts) - \phi(t)\,\zeta(s), \quad s > 0,$$

satisfy (3.13) too and, by (3.2), also the linear combinations

$$\psi(t) = \xi(t)\,\phi(s) + \eta(t)\,(s-1), \quad s > 0,$$

of functions $\xi(t)$ and $\eta(t)$ satisfy (3.13). Since $\phi(s)$ is not linear in the neighborhood of $s = 1$, it follows from here that $\xi(t)$ and $\eta(t)$ themselves satisfy (3.13). Therefore the integrals $\langle\xi\rangle$ and $\langle\eta\rangle$ exist and are finite. For the fixed alternatives $F \sim f$ we shall consider the linear combinations

$$\mu_\phi(f) = \langle\xi\rangle\, D_\phi(F_0, F) + \langle\phi\rangle\, D_\zeta(F_0, F)$$

of the integrals

$$D_\phi(F_0, F) = \int_0^1 f(x)\,\phi\left(\frac{f_0(x)}{f(x)}\right) \mathrm{d}x = \int_0^1 f(x)\,\phi\left(\frac{1}{f(x)}\right) \mathrm{d}x$$

and

$$D_\zeta(F_0, F) = \int_0^1 f(x)\,\zeta\left(\frac{f_0(x)}{f(x)}\right) \mathrm{d}x = \int_0^1 f(x)\,\zeta\left(\frac{1}{f(x)}\right) \mathrm{d}x$$

(cf. (1.6)) which are under the present assumptions about the alternative density f well defined and finite. If $\phi(t)$ is convex on $(0,\infty)$ or $\phi(t) - \phi'(1)\,(t-1)$ monotone on $(0,1)$ and $(1,\infty)$ then $D_\phi(F_0, F)$ is nonnegative ϕ-divergence or ϕ-disparity of F_0 and F. Similarly if $\zeta(t)$ is convex on $(0,\infty)$ or $\zeta(t) - \zeta(1) - \zeta'(1)\,(t-1)$ monotone on $(0,1)$ and $(1,\infty)$ then

$$D_{\phi^*}(F_0, F) = \int_0^1 f(x)\,\phi^*\left(\frac{f_0(x)}{f(x)}\right) \mathrm{d}x = D_\zeta(F_0, F) - 1$$

is the ϕ^*-divergence or ϕ^*-disparity of F_0 and F for

$$\phi^*(t) = \zeta(t) - \zeta(1) = \zeta(t) - 1 \quad \text{(cf. Lemmas 3.1 and 3.2).}$$

Hence the formula for $\mu_\phi(f)$ can be written for every $\phi \in \mathbf{\Phi}_1$ in the more intuitive form

$$\mu_\phi(f) = \langle\xi\rangle\, D_\phi(F_0, F) + \langle\phi\rangle\, [D_{\phi^*}(F_0, F) + 1] \tag{3.16}$$

where ξ and ϕ^* depend on ϕ as specified above and $D_\phi(F_0, F)$, $D_{\phi^*}(F_0, F)$ are divergences of disparities between the hypothesis F_0 and the alternative F for typical $\phi \in \mathbf{\Phi}_1$. For $\phi \in \mathbf{\Phi}_2 \subset \mathbf{\Phi}_1$ it holds $\phi^*(t) = t - 1$ so that the last formula simplifies as follows

$$\mu_\phi(f) = \langle\xi\rangle\, D_\phi(F_0, F) + \langle\phi\rangle. \tag{3.17}$$

In particular,

$$\mu_\phi(f_0) = \langle\phi\rangle. \tag{3.18}$$

Theorem 3.2. Consider the observations under the fixed alternative $F \sim f$ and denote by U_ϕ any statistic from the class $\{T_\phi, \tilde{T}_\phi, S_\phi, \tilde{S}_\phi\}$. If $\phi \in \mathbf{\Phi}_1$ satisfies (3.13) then

$$\frac{U_\phi}{n} \xrightarrow{p} \mu_\phi(f) \quad \text{for } n \to \infty \tag{3.19}$$

where $\mu_\phi(f)$ is given by (3.16). If $\phi \in \mathbf{\Phi}_2$ satisfies (3.13) then the asymptotic relation (3.19) remains valid also for $U_\phi = \tilde{T}_\phi^+$ and $U_\phi = S_\phi^+$ and $\mu_\phi(f)$ is given by the simpler formula (3.17).

Proof. By Theorem 1 of Hall (1984), the statistic $\tilde{S}_\phi$ defined by (2.13) satisfies under a fixed alternative $F \sim f$ the relation

$$\frac{\tilde{S}_\phi}{n} \xrightarrow{p} \tilde{\mu}_\phi(f) = \int_0^1 f^2(x) \left(\int_0^\infty \phi(t)\, e^{-tf(x)} \mathrm{d}t \right) \mathrm{d}x \quad \text{as } n \to \infty$$

provided $\phi : (0, \infty) \mapsto I\!R$ is continuous and exponentially bounded in the sense that $|\phi(t)| \leq K(t^\alpha + t^{-\beta})$ for some $K > 0$, $\alpha \geq 0$, $\beta < 1$ and f is bounded, piecewise continuous and bounded away from 0 (see also part (i) of Theorem 3.1 in Misra and van der Meulen (2001)). Thus (3.19) is proved for $U_\phi = \tilde{S}_\phi$ as soon as it is shown that for $\phi \in \mathbf{\Phi}_1$ the limit $\tilde{\mu}_\phi(f)$ coincides with $\mu_\phi(f)$. By substituting s for $tf(x)$ in the last integral and using the assumption $0 < f(x) < \infty$ and the functional equation (3.2),

$$\begin{aligned}
\tilde{\mu}_\phi(f) &= \int_0^1 f(x) \left(\int_0^\infty \phi\left(\frac{s}{f(x)}\right) e^{-s} \mathrm{d}s \right) \mathrm{d}x \\
&= \int_0^1 f(x) \left(\int_0^\infty \left[\xi(s)\, \phi\left(\frac{1}{f(x)}\right) + \zeta\left(\frac{1}{f(x)}\right) \phi(s) + \eta(s) \left(\frac{1}{f(x)} - 1\right) \right] e^{-s} \mathrm{d}s \right) \mathrm{d}x \\
&= \mu_\phi(f) + \int_0^\infty \eta(s)\, e^{-s}\, \mathrm{d}s \int_0^1 (1 - f(x))\, \mathrm{d}x = \mu_\phi(f).
\end{aligned}$$

The extension of (3.19) to $U_\phi \in \{T_\phi, \tilde{T}_\phi, S_\phi\}$ follows from Theorem 3.1. For $\phi \in \mathbf{\Phi}_2$ the extension of (3.19) to $U_\phi \in \{\tilde{T}_\phi^+, S_\phi^+\}$ follows from Theorem 3.1 too. □

In the sequel we use the L_2-norm

$$\|\ell\| = \left(\int_0^1 \ell^2(x)\, \mathrm{d}x \right)^{1/2}$$

and we usually denote the integral (3.14) by $\langle \psi(t) \rangle$ instead of $\langle \psi \rangle$.

Theorem 3.3. Consider the observations under the local alternatives with a limit function $\ell(x)$ of (3.8) and denote by U_ϕ any statistic from the set $\{T_\phi, \tilde{T}_\phi, \tilde{T}_\phi^+, S_\phi, \tilde{S}_\phi, S_\phi^+\}$. If $\phi \in \mathbf{\Phi}_2$ satisfies the stronger version of (3.13) with $\beta < 1/2$ then

$$\frac{1}{\sqrt{n}}(U_\phi - n\mu_\phi) \xrightarrow{\mathcal{D}} N(m_\phi(\ell), \sigma_\phi^2) \quad \text{a.s. } n \to \infty \tag{3.20}$$

where

$$\mu_\phi = \langle \phi(t) \rangle, \quad \sigma_\phi^2 = \langle \phi^2(t) \rangle - \langle \phi(t) \rangle^2 - (\langle t\phi(t) \rangle - \langle \phi(t) \rangle)^2 \tag{3.21}$$

and

$$m_\phi(\ell) = \frac{\|\ell\|^2}{2} \left(\langle t^2 \phi(t) \rangle - 4\langle t\phi(t) \rangle + 2\langle \phi(t) \rangle \right). \tag{3.22}$$

Proof. For $U_\phi = S_\phi^+$ the relations (3.20) – (3.22) follow from the result of Kuo and Rao (1981), cf. also Del Pino (1979) and Theorem 3.2 in Misra and van der Meulen (2001). The extension to the remaining statistics U_ϕ follows from Theorem 3.1. □

Let us now consider the fixed alternative $F \sim f$ defined in (i) above and $\phi \in \mathbf{\Phi}_2$ with $\xi = \xi_\phi$, $\eta = \eta_\phi$, satisfying the functional equation (3.3), and let $\tilde{f}(x) = [\phi'(1) - \eta'(1)]\, f(x) + \eta'(1)$ where ϕ', ξ', η' are the derivatives of ϕ, ξ, η. To express the asymptotic normality under this alternative, we need auxiliary functions $\Psi_i = \Psi_{i,\phi}$ of the variable $x \in (0,1)$:

$$\begin{aligned}
\Psi_1(x) &= \xi'(1)\,\langle\phi(t)\rangle\, f(x)\,\xi\left(\frac{1}{f(x)}\right) + \xi'(1)\, f(x)\,\phi\left(\frac{1}{f(x)}\right) + \tilde{f}(x),\\
\Psi_2(x) &= \left(\langle\phi^2(t)\rangle - \langle\phi(t)\rangle^2\right) f(x)\,\xi^2\left(\frac{1}{f(x)}\right) + f(x)\,\eta^2\left(\frac{1}{f(x)}\right)\\
&\quad +2(\langle t\phi(t)\rangle - \langle\phi(t)\rangle) f(x)\,\xi\left(\frac{1}{f(x)}\right)\eta\left(\frac{1}{f(x)}\right), \qquad (3.23)\\
\Psi_3(x) &= (\langle t\phi(t)\rangle - \langle\phi(t)\rangle)\sqrt{f(x)}\,\xi\left(\frac{1}{f(x)}\right) + \sqrt{f(x)}\,\eta\left(\frac{1}{f(x)}\right), \qquad (3.24)
\end{aligned}$$

and also

$$\Psi_4(x) = \frac{\sqrt{f(x)}}{F(x)} \int_0^x \left(1 - \frac{F(y)\, f'(y)}{f^2(y)}\right) \Psi_1(y)\,\mathrm{d}y \qquad (3.25)$$

when the alternative density has a continuous derivative $f'(x)$ on $(0,1)$.

Theorem 3.4. Consider the observations under the fixed alternative $F \sim f$ where f has a continuous derivative $f' : [0,1] \mapsto I\!R$, and denote by U_ϕ any statistic from the set $\{T_\phi, \tilde{T}_\phi, \tilde{T}_\phi^+, S_\phi, \tilde{S}_\phi, S_\phi^+\}$. If $\phi \in \mathbf{\Phi}_2$ satisfies the stronger version of (3.13) with $\beta < 1/2$ then

$$\frac{1}{\sqrt{n}}(U_\phi - n\mu_\phi(f)) \xrightarrow{\mathcal{D}} N(0, \sigma_\phi^2(f)) \quad n \to \infty \qquad (3.26)$$

where $\mu_\phi(f)$ is given by (3.17) and

$$\sigma_\phi^2(f) = \int_0^1 \Psi_2(x)\,\mathrm{d}x - 2\int_0^1 \Psi_3(x)\,\Psi_4(x)\,\mathrm{d}x + \int_0^1 \Psi_4^2(x)\,\mathrm{d}x \qquad (3.27)$$

for $\Psi_2(x)$, $\Psi_3(x)$ and $\Psi_4(x)$ defined by (3.23) – (3.25).

Proof. Consider $U_\phi = \tilde{S}_\phi$ for $\phi \in \mathbf{\Phi}_2$. By Lemma 3.2, $\phi(t)$ has a continuous derivative $\phi'(t)$ on $(0,\infty)$. By (3.5), for every $c \in I\!R$

$$t^c|\phi'(t)| \le |\xi'(1)|\, t^{c-1}|\phi(t)| + |\phi'(1)|\, t^c + |\eta'(1)|\, t^{c-1}|t-1|.$$

Thus if ϕ satisfies (3.13) with $\beta < 1/2$ then there exists $\alpha \geq 0$ such that

$$\lim_{t\to\infty} t^{-\alpha}|\phi'(t)| = \lim_{t\downarrow 0} t^{1+\beta}|\phi'(t)| = 0.$$

This means that under the assumptions of the theorem there exist $a > 0$, $K > 0$ and $b < 1/2$ such that for every $t \in (0, \infty)$

$$|\phi(t)| \leq K(t^a + t^{-b}) \quad \text{and} \quad |\phi'(t)| \leq K(t^a + t^{-b-1}).$$

For continuously differentiable functions ϕ satisfying these assumptions and fixed alternatives with densities f continuously differentiable on $(0, 1)$ it follows from Theorem 2 in Hall (1984) (cf. also part (ii) in Theorem 3.1 of Misra and van der Meulen (2001)) that $U_\phi = \tilde{S}_\phi$ satisfies the relation

$$\frac{1}{\sqrt{n}}(U_\phi - n\tilde{\mu}_\phi(f)) \xrightarrow{\mathcal{D}} N(0, \tilde{\sigma}^2_\phi(f)) \quad \text{for } n \to \infty$$

where: (1) the asymptotic mean $\tilde{\mu}_\phi(f)$ was presented and proved to be equal to $\mu_\phi(f)$ in the proof of Theorem 3.2 under assumptions weaker than here and, (2) the asymptotic variance $\tilde{\sigma}^2_\phi(f)$ can be specified by means of the standard exponential variate Z and the auxiliary function

$$G(x) = \int_0^x \left(1 - \frac{F(y)\, f'(y)}{f^2(y)}\right) E\left[Z\,\phi'\left(\frac{Z}{f(y)}\right)\right] \mathrm{d}y, \quad 0 < x < 1, \tag{3.28}$$

as the sum of

$$\begin{aligned} s_1^2(f) &= \int_0^1 \left(E\phi^2\left(\frac{Z}{f(x)}\right) - \left[E\phi\left(\frac{Z}{f(x)}\right)\right]^2\right) f(x)\,\mathrm{d}x \\ s_2^2(f) &= -2\int_0^1 E\left[(Z-1)\,\phi\left(\frac{Z}{f(x)}\right)\right] \frac{G(x)}{F(x)}\, f(x)\,\mathrm{d}x \end{aligned}$$

and

$$s_3^2(f) = \int_0^1 \left(\frac{G(x)}{F(x)}\right)^2 f(x)\,\mathrm{d}x.$$

It remains to be proved that for every $x \in (0, 1)$

$$\left(E\,\phi^2\left(\frac{Z}{f(x)}\right) - \left[E\,\phi\left(\frac{Z}{f(x)}\right)\right]^2\right) f(x) = \Psi_2(x), \tag{3.29}$$

$$E\left[(Z-1)\,\phi\left(\frac{Z}{f(x)}\right)\right] \sqrt{f(x)} = \Psi_3(x) \tag{3.30}$$

and

$$\frac{G(x)\sqrt{f(x)}}{F(x)} = \Psi_4(x). \tag{3.31}$$

Indeed, then $\tilde{\sigma}^2_\phi(t) = \sigma^2_\phi(f)$ so that (3.26) is proved for $U_\phi = \tilde{S}_\phi$ and the extension of (3.26) to the remaining statistics U_ϕ considered there follows from Theorem 3.1. We shall prove (3.29) – (3.31) in the reversed order. By substituting $t = Z/f(y)$ in (3.5) and taking into account that $\zeta(t) \equiv 1$ we obtain

$$\begin{aligned} E\left[Z\phi'\left(\frac{Z}{f(y)}\right)\right] &= f(y)\,E\left[\xi'(1)\,\phi\left(\frac{Z}{f(y)}\right) + \phi'(1) + \eta'(1)\left(\frac{Z}{f(y)} - 1\right)\right] \\ &= f(y)\left[\xi'(1)\,E\,\phi\left(\frac{Z}{f(y)}\right) + \phi'(1) + \eta'(1)\left(\frac{1}{f(y)} - 1\right)\right] \end{aligned}$$

and, by putting $s = 1/f(x)$ and $t = Z$ in (3.3), we get

$$\phi\left(\frac{Z}{f(x)}\right) = \phi(Z)\,\xi\left(\frac{1}{f(x)}\right) + \phi\left(\frac{1}{f(x)}\right) + \eta\left(\frac{1}{f(x)}\right)(Z-1). \tag{3.32}$$

Therefore

$$E\,\phi\left(\frac{Z}{f(x)}\right) = \langle\phi\rangle\xi\left(\frac{1}{f(x)}\right) + \phi\left(\frac{1}{f(x)}\right) \tag{3.33}$$

and, consequently,

$$E\left[Z\,\phi'\left(\frac{Z}{f(y)}\right)\right] = \Psi_1(y). \tag{3.34}$$

This together with the definitions of $\Psi_4(x)$ and $G(x)$ in (3.25) and (3.28) implies (3.31). Further, from (3.32) and the definition of $\Psi_3(x)$ in (3.24) we get (3.30). Finally, from (3.32), (3.33) and the definition of $\Psi_2(x)$ in (3.23) we obtain (3.29) which completes the proof. □

Remark 3.1. Under the hypothesis $F_0 \sim f_0 \equiv 1$ both Theorems 3.3 and 3.4 deal with the same statistical model. Therefore the asymptotic parameters $(\mu_\phi, \sigma^2_\phi)$ from (3.21) and $(\mu_\phi(f_0), \sigma^2_\phi(f_0))$ from (3.17) and (3.27) must be the same, i. e. the equalities

$$\mu_\phi(f_0) = \langle\phi\rangle \quad \text{and} \quad \sigma^2_\phi(f_0) = \langle\phi^2\rangle - \langle\phi\rangle^2 - (\langle t\phi(t)\rangle - \langle\phi\rangle)^2$$

must hold. The first equality is clear from (3.17), (3.18). For $f = f_0$ we get from (3.34) by partial integration

$$\Psi_1(y) = \langle t\phi'(t)\rangle = \langle t\phi(t)\rangle - \langle\phi\rangle \quad \text{for all } y \in (0,1).$$

Thus, by (3.25), $\Psi_4(x)$ is under the hypothesis constant, equal $\langle t\phi(t)\rangle - \langle\phi\rangle$. Similarly, by (3.23) (3.24) and Lemma 3.1, $\Psi_2(x) = \langle\phi^2\rangle - \langle\phi\rangle^2$ and $\Psi_3(x) = \Psi_4(x)$. Hence (3.27) implies the desired result

$$\sigma^2_\phi(f_0) = \Psi_2(x) - 2\Psi_4^2(x) + \Psi_4^2(x) = \sigma^2_\phi.$$

Remark 3.2. The expressions μ_ϕ, σ_ϕ^2 are well defined by (3.21) for every continuous function $\phi : (0,\infty) \mapsto I\!R$ satisfying the condition (3.13) with $\beta < 1/2$. If this condition holds for some function $\psi : (0,\infty) \mapsto I\!R$ then it holds also for all linear transformations $\phi(t) = a\psi(t) + b(t-1) + c$ and

$$\mu_\phi = a\mu_\psi + c, \quad \sigma_\phi^2 = a^2\sigma_\psi^2. \tag{3.35}$$

Let us now consider a fixed alternative $F \sim f$ with the density continuously differentiable on $(0,1)$. Then the formulas

$$\mu_\phi(f) = \int_0^1 f(x) \left\langle \phi\left(\frac{t}{f(x)}\right)\right\rangle \mathrm{d}x \quad \text{and} \quad \sigma_\phi^2(f) = s_1^2(f) + s_2^2(f) + s_3^2(f)$$

using $s_i^2(f)$ specified in the last proof, define $\mu_\phi(f)$ and $\sigma_\phi^2(f)$ for all continuously differentiable functions $\phi : (0,\infty) \mapsto I\!R$ such that both $\phi(t)$ and $\tilde{\phi}(t) = t\phi'(t)$ satisfy (3.13) with $\beta < 1/2$. If ψ is one of the functions satisfying all these conditions then all linear transformations $\phi(t) = a\psi(t) + b(t-1) + c$ satisfy these conditions too and

$$\mu_\phi(f) = a\mu_\psi(f) + c, \quad \sigma_\phi^2(f) = a^2\sigma_\psi^2(f). \tag{3.36}$$

The formulas (3.35) and (3.36) are verifiable from the definitions mentioned in this remark and they are useful for evaluation of asymptotic means and variances.

4 Asymptotic results for power divergence statistics

In this section we pay special attention to the class of convex functions $\phi_\alpha : (0,\infty) \mapsto I\!R$ parametrized by $\alpha \in I\!R$ and defined by

$$\phi_\alpha(t) = \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(\alpha-1)} \quad \text{if } \alpha \in I\!R - \{0,1\} \tag{4.1}$$

and otherwise by the corresponding limits

$$\phi_0(t) = -\ln t + t - 1 \quad \text{and} \quad \phi_1(t) = t\ln t - t + 1. \tag{4.2}$$

All these functions are strictly convex and arbitrarily differentiable on $(0,\infty)$ with $\phi_\alpha(1) = \phi_\alpha'(1) = 0$ and $\phi_\alpha''(1) = 1$. All of them belong to the subset $\mathbf{\Phi}_2 \subset \mathbf{\Phi}$, i. e. they satisfy the functional equation (3.3) with

$$\xi(t) = \xi_\alpha(t) = t^\alpha \quad \text{and} \quad \eta(t) = \eta_\alpha(t) = \begin{cases} \frac{t^\alpha - t}{\alpha - 1} & \text{if} \quad \alpha \neq 1 \\ \lim\limits_{\alpha\to 1} \frac{t^\alpha - t}{\alpha-1} = t\ln t & \text{if} \quad \alpha = 1 \end{cases} \tag{4.3}$$

i. e.

$$\phi_\alpha(st) = s^\alpha\phi_\alpha(t) + \phi_\alpha(s) + (t-1). \begin{cases} \frac{s^\alpha - s}{\alpha-1} & \text{if} \quad \alpha \neq 1 \\ s\ln s & \text{if} \quad \alpha = 1 \end{cases} \tag{4.4}$$

for all s, $t > 0$ and all parameters $\alpha \in I\!R$.

We use the simplified notation

$$D_\alpha(\boldsymbol{p}_0, \boldsymbol{p}) = D_{\phi_\alpha}(\boldsymbol{p}_0, \boldsymbol{p}) \quad \text{and} \quad D_\alpha(F_0, F) = D_{\phi_\alpha}(F_0, F)$$

for the ϕ_α-divergences. It is easy to see that

$$\tilde{\phi}_\alpha(t) = \frac{t^\alpha - 1}{\alpha(\alpha - 1)}, \quad \alpha \in I\!R - \{0, 1\}$$

and

$$\tilde{\phi}_0(t) = -\ln t, \quad \tilde{\phi}_1(t) = t \ln t$$

are convex functions belonging to $\boldsymbol{\Phi}_2$ too and that the $\tilde{\phi}_\alpha$-divergences coincide with the ϕ_α-divergences. Further,

$$\begin{aligned} D_2(\boldsymbol{p}_0, \boldsymbol{p}) &= D_{-1}(\boldsymbol{p}, \boldsymbol{p}_0) = \frac{1}{2}\chi^2(\boldsymbol{p}_0, \boldsymbol{p}), \\ D_1(\boldsymbol{p}_0, \boldsymbol{p}) &= D_0(\boldsymbol{p}, \boldsymbol{p}_0) = I(\boldsymbol{p}_0, \boldsymbol{p}), \\ D_{1/2}(\boldsymbol{p}_0, \boldsymbol{p}) &= D_{1/2}(\boldsymbol{p}, \boldsymbol{p}_0) = 4H(\boldsymbol{p}_0, \boldsymbol{p}). \end{aligned}$$

Similar equalities hold also when $\boldsymbol{p}_0$, $\boldsymbol{p}$ are replaced by F_0, F. We see from here that the class of statistics $T_\phi = nD_\phi(\boldsymbol{p}_0, \boldsymbol{p})$ with $\phi \in \{\phi_\alpha : \alpha \in I\!R\}$ contains the classical statistics (2.1) – (2.6) as particular cases and thus provides a sufficient wide variety of statistics for theoretical and practical considerations.

In this section we study the sets of ϕ_α-divergence statistics

$$\mathcal{U}_\alpha = \left\{T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}, \tilde{T}^+_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, S^+_{\phi_\alpha}\right\} \tag{4.5}$$

for $\alpha \in I\!R$. The statistics T_{ϕ_α}, $\tilde{T}_{\phi_\alpha}$ and $S^+_{\phi_\alpha}$ are not altered if the nonnegative convex functions $\phi_\alpha \in \boldsymbol{\Phi}_2$ are replaced by the simpler convex functions $\tilde{\phi}_\alpha \in \boldsymbol{\Phi}_2$. The statistics T_{ϕ_α} and $\tilde{T}_{\phi_\alpha}$ are proportional to the ϕ_α-divergences of hypothetical and empirical distributions F_0 and F_n reduced by appropriate partitions of the observation space $[0, 1]$. For the remaining statistics from $\mathcal{U}_\alpha$ one cannot find partitions of $[0, 1]$ enabling such a ϕ_α-divergence interpretation but these statistics still reflect a proximity of F_0 and F reduced by some partitions, using the functions ϕ_α or $\tilde{\phi}_\alpha$. Among them are the spacings-based statistics studied in the previous literature.

For example

$$\sum_{j=1}^{n+1}(Y_j - Y_{j-1})^2 = \frac{1}{n+1}\left(1 + \frac{2S^+_{\phi_2}}{n+1}\right) = \frac{1}{n+1}\left(1 + \frac{2S^+_{\tilde{\phi}_2}}{n+1}\right)$$

with $Y_0 = 0$, $Y_{n+1} = 1$ is the so-called Greenwood statistic introduced by Greenwood (1946) and studied later by Moran (1951) and many others. The statistic $S^+_{\phi_0} = S^+_{\tilde{\phi}_0}$ was introduced by Moran (1951) and studied later by Cressie (1976), van Es (1992), Ekström (1999) and many others cited by them. A class of statistics containing $\{\tilde{S}_{\phi_\alpha} : \alpha > -1/2\}$ was studied by Hall (1984) and a class containing $\{\tilde{T}^+_{\tilde{\phi}_\alpha} : \alpha \in I\!R\}$ or $\{S_{\phi_\alpha} : \alpha \in I\!R\}$ by Hall (1986) or Jammalamadaka et al (1986, 1989), respectively. Recently Misra and van der Meulen (2001) investigated the statistic $S^+_{\phi_1} = S^+_{\tilde{\phi}_1}$ (including its generalization to the m-spacings for fixed $m > 1$). The only paper dealing so far with the spacings-based statistics with a direct ϕ_α-divergence interpretation seems to be that of Morales et al (2003) which studies a class of statistics containing $\{\tilde{T}_{\phi_\alpha} : \alpha \in I\!R\}$, but the asymptotic theory is restricted there to the m-spacings with $m = m_n$ increasing to infinity for $n \to \infty$, similarly as in Hall (1986) or Jammalamadaka et al (1986, 1989).

Since the asymptotic theory of the statistics $U_\alpha \in \mathcal{U}_\alpha$ specified by (4.5) is covered by Theorems 3.1 − 3.4, the theorems that follow are their corollaries. However, the proofs of the following theorems are partly based on a continuity theory for the asymptotic parameters

$$\mu_\alpha(f) = \mu_{\phi_\alpha}(f), \quad \sigma^2_\alpha(f) = \sigma^2_{\phi_\alpha}(f), \quad \mu_\alpha = \mu_{\phi_\alpha}, \quad \sigma^2_\alpha = \sigma^2_{\phi_\alpha} \quad \text{and} \quad m_\alpha(\ell) = m_{\phi_\alpha}(\ell) \tag{4.6}$$

as functions of the structural parameter $\alpha \in I\!R$. This theory enables us to avoid a direct calculation of the asymptotic parameters at some $\alpha_0 \in I\!R$ if these calculations are tedious and the asymptotic parameters are known at the neighbors α of α_0. This theory is summarized in Theorem 4.1 using the following lemma.

Lemma 4.1. Let $g(y)$ be a continuous positive function on a compact interval $[a, b] \subset I\!R$ and $\Phi(u, v)$ a continuous function of variables $u, v \in I\!R$. Further, let for all α from an interval $(c, d) \subset I\!R$, $\psi_\alpha : (0, \infty) \mapsto I\!R$ be convex or concave functions differentiable at some point $t_* \in (0, \infty)$. If the values $\psi_\alpha(t)$, $t \in (0, \infty)$ and the derivatives $\psi'_\alpha(t_*)$ continuously depend on $\alpha \in (c, d)$ then for every $\alpha_0 \in (c, d)$

$$\lim_{\alpha \to \alpha_0} \int_a^b \Phi(g, \psi_\alpha(g))\, \mathrm{d}y = \int_a^b \Phi(g, \psi_{\alpha_0}(g))\, \mathrm{d}y. \tag{4.7}$$

Proof. By the assumptions about g,

$$t_0 = \min_{y \in [a,b]} g(y) > 0 \quad \text{and} \quad t_1 = \max_{y \in [a,b]} g(y) < \infty.$$

If $\psi_\alpha(t)$ is convex then for every $t \in [t_0, t_1]$ and $\alpha \in (c, d)$

$$\psi'_\alpha(t_*)\,(t - t_*) \le \psi_\alpha(t) \le \psi_\alpha(t_0) + \psi_\alpha(t_1).$$

If $\psi_\alpha(t)$ is concave then, similarly,

$$\psi_\alpha(t_0) + \psi_\alpha(t_1) \le \psi_\alpha(t) \le \psi'_\alpha(t_*)\,(t - t_*).$$

Therefore in both cases

$$\max_{t_0 \le t \le t_1} |\psi_\alpha(t)| \le \max\left\{|\psi_\alpha(t_0) + \psi_\alpha(t_1)|,\, |\psi'_\alpha(t_*)| \cdot |t_1 - t_0|\right\}.$$

The assumed continuity of $\psi'_\alpha(t_*)$ and $\psi_\alpha(t_0) + \psi_\alpha(t_1)$ in the variable $\alpha \in (c, d)$ implies that for all compact neighborhoods $N \subset (c, d)$ of α_0 the constant

$$k = \sup_{\alpha \in N} \max_{t_0 \le t \le t_1} |\psi_\alpha(t)| = \sup_{\alpha \in N} \max_{y \in [a,b]} |\psi_\alpha(g(y))|$$

is finite. Put

$$K = \max_{[t_0,t_1]\times[-k,k]} \Phi(u, v).$$

The function $|\Phi(g, \psi_\alpha(g))|$ of variables $(y, \alpha) \in [a, b] \times (c, d)$ is bounded on $[a, b] \times N$ by $K < \infty$. Since for every $y \in [a, b]$

$$\lim_{\alpha \to \alpha_0} \Phi(g, \psi_\alpha(g)) = \Phi(g, \psi_{\alpha_0}(g)),$$

the Lebesgue dominated convergence theorem for integrals implies (4.7). □

Theorem 4.1. The asymptotic parameters μ_α, σ^2_α and $m_\alpha(\ell)$ specified by (4.6) and (3.21), (3.22) are continuous in the variable $\alpha \in (-1/2, \infty)$. If the density f satisfies the assumptions of Theorem 3.2 then the asymptotic mean $\mu_\alpha(f)$ specified by (4.6) and (3.17) is continuous in the variable $\alpha \in (-1, \infty)$. If the density f satisfies the assumptions of Theorem 3.4 then the asymptotic variance $\sigma^2_\alpha(f)$ specified by (4.6) and (3.27) is continuous in the variable $\alpha \in (-1/2, \infty)$.

Proof. Since $\mu_\alpha = \mu_\alpha(f_0)$ and $\sigma^2_\alpha = \sigma^2_\alpha(f_0)$ where the hypothetic density f_0 satisfies the assumptions of Theorems 3.2 and 3.4, the continuity of μ_α and σ^2_α follows from the continuity of $\mu_\alpha(f)$ and $\sigma^2_\alpha(f)$ proved below. By (4.6) and (3.22),

$$m_\alpha(\ell) = \frac{\|\ell\|^2}{2}\Big(\langle t^2\phi_\alpha(t)\rangle - 4\langle t\phi_\alpha(t)\rangle + 2\langle\phi_\alpha(t)\rangle\Big)$$

where ϕ_α is given by (4.1) (4.2) and, by (3.14),

$$\langle t^j\phi_\alpha(t)\rangle = \int_0^\infty t^j\phi_\alpha(t)\,\mathrm{d}G(t), \quad j \in \{0, 1, 2\} \tag{4.8}$$

for $G(t) = 1 - e^{-t}$. All integrals (4.8) are finite if and only if $\alpha \in (-1, \infty)$. Further, for every fixed $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,\alpha\phi_\alpha(t) \ge 0 \quad \text{at any } \alpha \in I\!R. \tag{4.9}$$

Hence the continuity of the products $\alpha\langle t^j\phi_\alpha(t)\rangle$ in the variable $\alpha \in I\!R$ follows from the monotone convergence theorem for integrals, and this implies also the desired continuity of the integrals (4.8) at any $\alpha \in (-1,\infty) - \{0\}$. Further, for every fixed $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,(\alpha-1)\,\phi_\alpha(t) \geq 0 \quad \text{for any } \alpha \in I\!R. \tag{4.10}$$

Hence the continuity of the products $(\alpha-1)\,\langle t^j\phi_\alpha(t)\rangle$ in the variable $\alpha \in I\!R$ follows from the monotone convergence theorem for integrals. Similarly as above, this implies the continuity of the integrals (4.8) at the remaining point $\alpha = 0$. Further, by (4.6) and (3.17),

$$\mu_\alpha(f) = \langle \xi_\alpha\rangle\, D_\alpha(F_0,F) + \langle \phi_\alpha\rangle$$

where, by (3.14) and (4.3)

$$\langle \xi_\alpha\rangle = \int_0^\infty t^\alpha \mathrm{d}G(t) \quad \text{and} \quad \langle \phi_\alpha\rangle = \int_0^\infty \phi_\alpha(t)\,\mathrm{d}G(t).$$

These integrals are finite if and only if $\alpha \in (-1,\infty)$. The continuity of $\langle\phi_\alpha\rangle$ at $\alpha \in (-1,\infty)$ was proved above, the continuity of $D_\alpha(F_0,F)$ at $\alpha \in I\!R$ follows from the assumptions about the densities f_0, f and from Proposition 2.14 in Liese and Vajda (1987). The continuity of $\langle\xi_\alpha\rangle$ of $\alpha \in (-1,\infty)$ follows from the monotone convergence theorem for integrals applied separately to the integration domains $(0,1)$ and $(1,\infty)$. Finally, let us consider $\sigma^2_\alpha(f)$ defined by (3.23)–(3.27) for $\phi = \phi_\alpha$, $\xi = \xi_\alpha$ and $\eta = \eta_\alpha$ given by (4.1)–(4.3). The integrals $\langle t\phi_\alpha(t)\rangle$, $\langle\phi_\alpha(t)\rangle$ and $\langle\phi^2_\alpha(t)\rangle$ are finite if and only if $\alpha \in (-1/2,\infty)$ and their continuity at $\alpha \in (-1/2,\infty)$ was either proved above or it can be proved similarly as above. The continuity of the integral

$$\int_0^1 \left[f\xi^2_\alpha\left(\frac{1}{f}\right) + f\eta^2_\alpha\left(\frac{1}{f}\right)\right] \mathrm{d}x$$

at $\alpha \in (-1/2,\infty)$ follows from Lemma 4.1, which classifies the continuity of the component $\int \psi_2(x)\,\mathrm{d}x$ of $\sigma^2_\alpha(f)$ in (3.27). For the continuity of the remaining two components take into account that $F(x) > c_1 x$ for some $c_1 > 0$ on $[0,1]$ because f is bounded away from zero on $[0,1]$. Further, both $f(x)$ and $f'(x)$ are bounded on $[0,1]$ so that there exists a constant c_2 such that in (3.25)

$$\frac{\sqrt{f(x)}}{F(x)} \int_0^x \left| 1 - \frac{F(y)\,f'(y)}{f^2(y)} \right| \mathrm{d}y < c_2 \quad \text{for all } x \in [0,1]. \tag{4.11}$$

Using the function $\varphi_\alpha(t) = \alpha\phi_\alpha(t)$ which is for every $t > 0$ continuous and monotone in $\alpha \in I\!R$ (cf. (4.9)), we obtain from (3.23)

$$\Psi_1(x) = \alpha\langle\phi_\alpha\rangle\, f(x)^{1-\alpha} + f(x)\,\varphi_\alpha\left(\frac{1}{f(x)}\right) + 1 - f(x)$$

where the right-hand side is bounded on $[0,1]$ locally uniformly in α and continuous at any $\alpha \in I\!R$. By (3.25) and (4.11), this implies that also $\Psi_4(x)$ is bounded on $[0,1]$ locally uniformly in α and continuous at any $\alpha \in I\!R$. Since the integrands in

$$\int_0^1 \left[\sqrt{f}\xi_\alpha\left(\frac{1}{f}\right) + \sqrt{f}\eta_\alpha\left(\frac{1}{f}\right)\right]\Psi_4\,\mathrm{d}x \quad \text{and} \quad \int_0^1 \Psi_4^2\,\mathrm{d}x$$

are on $[0,1]$ continuous and locally bounded in the variable $\alpha \in I\!R$, the continuity of both these integrals in the variable $\alpha \in I\!R$ follows from the Lebesgue dominated convergence theorem for integrals. This clarifies the continuity of the second and third component of $\sigma_\alpha^2(f)$ in (3.27) and thus completes the proof. □

In the theorems below we use the gamma function of the variable $\alpha \in I\!R$ and the Euler constant,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\,e^{-t}\,\mathrm{d}t \quad \text{and} \quad \gamma = 0.577\ldots. \tag{4.12}$$

Theorem 4.2. Consider the observations under the fixed alternative $F \sim f$ and denote by U_α any statistic from the class $\mathcal{U}_\alpha$ of (4.5). If $\alpha > -1$ then

$$\frac{U_\alpha}{n} \xrightarrow{p} \mu_\alpha(f) \quad \text{as } n \to \infty \tag{4.13}$$

for

$$\mu_\alpha(f) = D_\alpha(F_0, F)\,\Gamma(\alpha+1) + \mu_\alpha, \tag{4.14}$$

where

$$\mu_0 = \gamma, \quad \mu_1 = 1-\gamma \quad \text{and} \quad \mu_\alpha = \frac{\Gamma(\alpha+1)-\Gamma(1)}{\alpha(\alpha-1)} \quad \text{for } \alpha \notin \{0,1\} \tag{4.15}$$

and $D_\alpha(F_0, F)$ are the ϕ_α-divergences

$$D_1(F_0,F) = \int_0^1 f\ln\frac{f}{f_0}\,\mathrm{d}x = \int_0^1 f(x)\ln f(x)\,\mathrm{d}x,$$

$$D_0(F_0,F) = \int_0^1 f_0\ln\frac{f_0}{f}\,\mathrm{d}x = -\int_0^1 \ln f(x)\,\mathrm{d}x, \tag{4.16}$$

$$D_\alpha(F_0,F) = \frac{1}{\alpha(\alpha-1)}\left(\int_0^1 f\left(\frac{f_0}{f}\right)^\alpha \mathrm{d}x - 1\right) = \frac{1}{\alpha(\alpha-1)}\left(\int_0^1 f(x)^{1-\alpha}\mathrm{d}x - 1\right) \quad \text{for } \alpha \notin \{0,1\}. \tag{4.17}$$

The ϕ_α-divergences are zero if and only if $F = F_0$ so that under the hypothesis $F = F_0$

$$\mu_\alpha(f_0) = \mu_\alpha, \quad \alpha \in I\!R. \tag{4.18}$$

Both parameters μ_α and $\mu_\alpha(f)$ are continuous in the variable $\alpha \in (-1,\infty)$ and satisfying the inequality $\mu_\alpha(f) \geq \mu_\alpha$ which is strict unless $F = F_0$.

Proof. The functions from the class $\{\phi_\alpha : \alpha \in (-1,\infty)\} \subset \mathbf{\Phi}_2$ satisfy all assumptions of Theorem 3.2. Hence (4.13) holds for all $\alpha > -1$ and the limit $\mu_\alpha(f)$ is given in accordance with (3.17) and (4.3) by the formula

$$\mu_\alpha(f) = \langle \xi_\alpha(t)\rangle \, D_\alpha(F_0, F) + \langle \phi_\alpha(t)\rangle = \langle t^\alpha\rangle \, D_\alpha(F_0, F) + \langle \tilde{\phi}_\alpha(t)\rangle$$

where $\langle t^\alpha\rangle = \Gamma(\alpha+1)$ for all $\alpha \in I\!\!R$. If $\alpha \notin \{0,1\}$ then

$$\langle \tilde{\phi}_\alpha(t)\rangle = \frac{1}{\alpha(\alpha-1)}\langle t^\alpha - 1\rangle = \frac{\Gamma(\alpha+1)-\Gamma(1)}{\alpha(\alpha-1)}$$

but

$$\langle \tilde{\phi}_0(t)\rangle = \langle -\ln t\rangle \quad \text{and} \quad \langle \tilde{\phi}_1(t)\rangle = \langle t\ln t\rangle$$

leads to evaluation of unpleasant integrals. This evaluation can be avoided by employing Theorem 4.1. By the continuity of $\mu_\alpha = \langle \tilde{\phi}_\alpha(t)\rangle$,

$$\mu_j = \langle \tilde{\phi}_j(t)\rangle = \lim_{\alpha\to j} \frac{\Gamma(\alpha+1)-\Gamma(1)}{\alpha(\alpha-1)} \quad \text{for } j \in \{0,1\},$$

where the limit on the right leads to the values μ_j, $j \in \{0,1\}$ given in (4.15), e.g. by using the L'Hospital rule and the known formulas $\Gamma'(1) = -\gamma$, $\Gamma'(2) = 1-\gamma$. The continuity and the inequality $\mu_\alpha(f) \geq \mu_\alpha$ for $\alpha \in (-1,\infty)$ follow from (4.14) and (4.15) because $D_\alpha(F_0,F)$ is nonnegative and continuous in $\alpha \in I\!\!R$ and $\Gamma(\alpha+1)$ is positive and continuous in $\alpha \in (-1,\infty)$. The condition for equality follows from the fact that $D_\alpha(F_0,F)$ is positive unless $F = F_0$. □

Since $\Gamma(\alpha+1) = \alpha(\alpha-1)\,\Gamma(\alpha-1)$, (4.14) and (4.15) can be replaced for $\alpha \notin \{0,1\}$ by

$$\mu_\alpha = \Gamma(\alpha-1) - \frac{1}{\alpha(\alpha-1)} \quad \text{and} \quad \mu_\alpha(f) = \Gamma(\alpha-1)\int_0^1 f^{1-\alpha}\,\mathrm{d}x - \frac{1}{\alpha(\alpha-1)}. \tag{4.19}$$

Theorem 4.2 can be illustrated by Table 4.1 presenting actual values of the parameters μ_α and $\mu_\alpha(f)$ for selected parameters α. In this table f denotes any density considered in Theorem 4.2.

Table 4.1 Values of μ_α and $\mu_\alpha(f)$ for selected $\alpha > -1$.

α	μ_α	$\mu_\alpha(f)$
$-\frac{1}{2}$	$\frac{4}{3}(\sqrt{\pi}-1) \doteq 1.030$	$\sqrt{\pi}\, D_{-1/2}(F_0,F) + \mu_{-1/2} = \frac{4\sqrt{\pi}}{3}\int_0^1 f^{3/2}\,\mathrm{d}x - \frac{4}{3}$
0	$\gamma \doteq 0.577$	$I(F,F_0) + \mu_0 = \int_0^1 f\ln f\,\mathrm{d}x + \gamma$
$\frac{1}{2}$	$4-2\sqrt{\pi} \doteq 0.455$	$2\sqrt{\pi}\, H(F_0,F) + \mu_{1/2} = 4 - 2\sqrt{\pi}\int_0^1 \sqrt{f}\mathrm{d}x$
1	$1-\gamma \doteq 0.423$	$I(F_0,F) + \mu_1 = 1 - \gamma - \int_0^1 \ln f\,\mathrm{d}x$
$\frac{3}{2}$	$\sqrt{\pi} - \frac{4}{3} \doteq 0.439$	$\frac{3\sqrt{\pi}}{4} D_{3/2}(F_0,F) + \mu_{3/2} = \sqrt{\pi}\int_0^1 \frac{\mathrm{d}x}{\sqrt{f}} - \frac{4}{3}$
2	$\frac{1}{2}$	$\chi^2(F_0,F) + \mu_2 = \int_0^1 \frac{\mathrm{d}x}{f} - \frac{1}{2}$
$\frac{5}{2}$	$\frac{\sqrt{\pi}}{2} - \frac{4}{15} \doteq 0.620$	$\frac{15\sqrt{\pi}}{8} D_{5/2}(F_0,F) + \mu_{5/2} = \frac{\sqrt{\pi}}{2}\int_0^1 \frac{\mathrm{d}x}{f^{3/2}} - \frac{4}{15}$
3	$\frac{5}{6} \doteq 0.833$	$6D_3(F_0,F) + \mu_3 = \int_0^1 \frac{\mathrm{d}x}{f^2} - \frac{1}{6}$
4	$\frac{23}{12} \doteq 1.917$	$24D_4(F_0,F) + \mu_4 = 2\int_0^1 \frac{\mathrm{d}x}{f^3} - \frac{1}{12}$

Theorem 4.3. Consider the observations under the local alternatives with the limit function $\ell(x)$ of (3.8) and denote by U_α any statistic from the class $\mathcal{U}_\alpha$ of (4.5). If $\alpha > -1/2$ then

$$\frac{1}{\sqrt{n}}(U_\alpha - n\mu_\alpha) \xrightarrow{\mathcal{D}} N(m_\alpha(\ell), \sigma_\alpha^2) \quad \text{as } n \to \infty \tag{4.20}$$

where the parameters μ_α, $m_\alpha(\ell)$ and σ_α^2 are continuous in the variable $\alpha \in (-1/2, \infty)$, given by (4.15) and by the formulas

$$m_\alpha(\ell) = \frac{\|\ell\|^2}{2}\,\Gamma(\alpha+1) \tag{4.21}$$

and

$$\sigma_0^2 = \frac{\pi^2}{6} - 1, \quad \sigma_1^2 = \frac{\pi^3}{3} - 3, \quad \sigma_\alpha^2 = \frac{\Gamma(2\alpha+1) - (\alpha^2+1)\,\Gamma^2(\alpha+1)}{\alpha^2(\alpha-1)^2} \quad \text{for } \alpha \notin \{0,1\}. \tag{4.22}$$

Proof. Similarly as in the previous proof, (4.20) follows for all $\alpha > -1/2$ from Theorem 3.3. If $\alpha \notin \{0,1\}$ then the expressions for $m_\alpha(\ell)$ and σ_α^2 given in (4.21) and (4.22) follow easily from the formulas given for $m_{\phi_\alpha}(\ell)$ and $\sigma_{\phi_\alpha}^2$ in Theorem 3.3. The direct evaluation of $m_j(\ell)$ and σ_j^2 from these formulas for $j \in \{0,1\}$ is a somewhat tedious task.

But using the continuity of $m_\alpha(\ell)$ and σ_α^2 established in Theorem 4.1, we obtain $m_j(\ell)$ and σ_j^2 given in (4.21) and (4.22) as the limits

$$m_j(\ell) = \lim_{\alpha \to j} m_\alpha(\ell) \quad \text{and} \quad \sigma_j^2 = \lim_{\alpha \to j} \sigma_\alpha^2 \quad \text{for } j \in \{0, 1\},$$

by using the continuity of the right-hand side of (4.21) and the L'Hospital rule, employing the formulas

$$\begin{aligned} \Gamma(\alpha + k + 1) &= (\alpha + k)\,(\alpha + k - 1) \cdots (\alpha + 1)\,\Gamma(\alpha + 1), \\ \Gamma''(\alpha + 1) &= 2\Gamma'(\alpha) + \alpha\Gamma''(\alpha) \end{aligned}$$

and

$$\Gamma''(1) = \frac{\pi^2}{6} + \gamma^2, \quad \Gamma''(2) = \frac{\pi^2}{6} - 2\gamma + \gamma^2$$

in addition to $\Gamma'(1), \Gamma'(2)$ given above. □

Theorem 4.3 provides a possibility to compare asymptotic relative efficiencies of the tests of hypothesis $\mathcal{H}_0 : F_0 \sim f_0$ based on the statistics $U_\alpha \in \mathcal{U}_\alpha$, $\alpha > -1/2$. The Pitman asymptotic relative efficiency (ARE) of one test relative to another is defined as the limit of the inverse ratio of sample sizes required to obtain the same limiting power at the sequence of alternatives converging to the null hypothesis. If we define the "efficacies" of the statistics $U_\alpha \in \mathcal{U}_\alpha$ of Theorem 4.3 by

$$\text{eff}(U_\alpha) = \frac{\Gamma^2(\alpha + 1)}{\sigma_\alpha^2} = \frac{(m_\alpha(\ell))^2}{\sigma_\alpha^2} \left(\frac{2}{\|\ell\|^2} \right)^2 \quad \text{for } \|\ell\|^2 \neq 0$$

then at the sequences of alternatives (3.7)

$$\text{ARE}(U_{\alpha_1}, U_{\alpha_2}) = \frac{\text{eff}(U_{\alpha_1})}{\text{eff}(U_{\alpha_2})}$$

(cf. Section 4 in Del Pino (1979)) where U_{α_1} and U_{α_2} are arbitrary statistics from $\mathcal{U}_{\alpha_1}$ and $\mathcal{U}_{\alpha_2}$. In Table 4.2 we present the parameters $m_\alpha(\ell)$, σ_α^2 and $\Gamma^2(\alpha+1)/\sigma_\alpha^2$ for selected values of $\alpha > -1/2$. Table 4.2 indicates that the statistics $U_2 \in \{T_{\phi_2}, \tilde{T}_{\phi_2}, \tilde{T}_{\phi_2}^+, S_{\phi_2}, \tilde{S}_{\phi_2}, S_{\phi_2}^+\}$ are most asymptotically efficient in the Pitman sense among all statistics U_α, $\alpha > -1/2$. This extends the result about the asymptotic efficiency of the Greenwood statistics $(2S_{\phi_2}^+ + n + 1)/(n + 1)^2$ (see the discussion at the end of this section) on p. 1457 in Rao and Kuo (1984).

Table 4.2 The asymptotic parameters $m_\alpha(\ell)$, σ^2_α and $\mathrm{eff}(U_\alpha)$

for selected statistics U_α of Theorem 4.3.

α	$m_\alpha(\ell)$	σ^2_α	$\mathrm{eff}(U_\alpha)$
0	$\frac{\Vert\ell\Vert^2}{2}$	$\frac{\pi^2}{6}-1 \doteq 0.645$	1.550
$\frac{1}{2}$	$\Vert\ell\Vert^2\,\frac{\sqrt{\pi}}{4} \doteq \frac{\Vert\ell\Vert^2}{2}\times 0.886$	$16-5\pi \doteq 0.292$	2.690
1	$\frac{\Vert\ell\Vert^2}{2}$	$\frac{\pi^2}{3}-3 \doteq 0.290$	3.448
$\frac{3}{2}$	$\Vert\ell\Vert^2\,\frac{3\sqrt{\pi}}{8} \doteq 1.329$	$\frac{32}{3}-\frac{13\pi}{4} \doteq 0.457$	3.871
2	$\Vert\ell\Vert^2 = \frac{\Vert\ell\Vert^2}{2}\times 2$	1	4.000
$\frac{5}{2}$	$\Vert\ell\Vert^2\,\frac{15\sqrt{\pi}}{16} \doteq \frac{\Vert\ell\Vert^2}{2}\times 3.323$	$\frac{128}{15}-\frac{29\pi}{16} \doteq 2.839$	3.890
3	$\Vert\ell\Vert^2\, 3 = \frac{\Vert\ell\Vert^2}{2}\times 6$	10	3.600
4	$\Vert\ell\Vert^2\, 12 = \frac{\Vert\ell\Vert^2}{2}\times 24$	212	2.717

The general form of the asymptotic normality (4.20) as well as the continuity of the parameters μ_α, $m_\alpha(\ell)$ and σ^2_α in $\alpha \in (-1/2,\infty)$ established in Theorem 4.3 seem to be new results. The special result for $\alpha = 0$ seems also be new. The particular result for $\alpha \in (-1/2,\infty) - \{0,1\}$ and $U_\alpha = S^+_{\phi_\alpha}$ follows from the asymptotic normality obtained for the statistics

$$\sum_{j=1}^{n+1} \left((n+1)\,(Y_j - Y_{j-1})\right)^\alpha = \alpha(\alpha-1)\, S^+_{\phi_\alpha} + n + 1$$

by Del Pino, see p. 1062 in Del Pino (1979). The particular result for $\alpha = 1$ and the statistics $U_1 = S^+_{\phi_1}$ with μ_1 and σ^2_1 given in the Tables 4.1 and 4.2 was obtained recently by Misra and van der Meulen (2001) who however considered m-spacings for arbitrary $m \geq 1$. They compared also the efficiency of the test statistics for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$ with a similar conclusion as in the Table 4.2.

In the rest of this section we consider the observations under the fixed alternative $F \sim f$ where f has a continuous derivative $f' : [0,1] \mapsto I\!R$ and denote by U_α any statistic from the set $\mathcal{U}_\alpha$ of (4.5). The functions from the class $\{\phi_\alpha : \alpha \in (-1/2,\infty)\}$ satisfy the assumption of Theorem 3.4. Therefore if $\alpha > -1/2$ then Theorem 3.4 implies that

$$\frac{1}{\sqrt{n}}(U_\alpha - n\mu_\alpha(f)) \xrightarrow{\mathcal{D}} N(0,\sigma^2_\alpha(f)) \quad \text{for } n \to \infty \tag{4.23}$$

where the asymptotic parameters $\mu_\alpha(f)$, $\sigma^2_\alpha(f)$ are given by (4.6). Similarly as in the previous two theorems, we are interested in explicit formulas for these parameters. By Theorem 3.4, the asymptotic mean is for all $\alpha \in I\!R$ given by the explicit formula presented in Theorem 4.2. The only problem which remains is the formula for $\sigma^2_\alpha(f)$, $\alpha \in I\!R$.

The functions $\psi_\alpha(t) = t^\alpha$ with $\alpha > -1/2$ satisfy all assumptions of Remark 3.2 so that we can consider the quantities

$$s_\alpha^2(f) = \sigma_{\psi_\alpha}^2(f), \quad \alpha \in (-1/2, \infty)$$

defined there. By (3.36),

$$\sigma_\alpha^2(f) = \frac{s_\alpha^2(f)}{\alpha^2(\alpha-1)^2} \quad \text{for } \alpha \in (-1/2,\infty) - \{0,1\}. \tag{4.24}$$

For $s_\alpha^2(f)$ and all $\alpha \in (-1/2,\infty) - \{0,1\}$ we can find on p. 521 of Hall (1984) an expression which can be given the form

$$s_\alpha^2(f) = \alpha^2(\alpha-1)^2 \left(\sigma_\alpha^2 \int_0^1 f^{1-2\alpha} \mathrm{d}x + \Gamma^2(\alpha+1)\, \Delta_\alpha(F_0, F) \right) \tag{4.25}$$

for σ_α^2 defined by the formula of (4.22) corresponding to $\alpha \notin \{0,1\}$ and

$$\Delta_\alpha(F_0,F) = \frac{1}{\alpha^2} \int_0^1 \left(\frac{1}{(f(x))^\alpha} - \frac{1}{F(x)} \int_0^x (f(y))^{1-\alpha} \mathrm{d}y \right)^2 f(x)\, \mathrm{d}x \quad \text{for } \alpha \in I\!R - \{0\}. \tag{4.26}$$

Since Hall (1984) gave no hint about derivation of his formula, let us mention that (4.25) is obtained if we substitute ψ_α for ϕ in $s_j^2(f)$, $j \in \{1,2,3\}$ from the proof of Theorem 3.4, and then employ the expression

$$\begin{aligned} G(x) &= \alpha E(Z^\alpha) \int_0^x \left(1 - \frac{F f'}{f^2} \right) \frac{1}{f^{\alpha-1}} \mathrm{d}y \\ &= \Gamma(\alpha+1) \left((\alpha-1) \int_0^x (f(y))^{1-\alpha} \mathrm{d}y + (f(x))^{-\alpha} F(x) \right) \end{aligned}$$

for $G(x)$ of (3.28). By (4.24) and (4.25),

$$\sigma_\alpha^2(f) = \sigma_\alpha^2 \int_0^1 f^{1-2\alpha} \mathrm{d}x + \Gamma^2(\alpha+1)\, \Delta_\alpha(F_0,F), \quad \alpha \in (-1/2,\infty) - \{0,1\}.$$

The final intuitively appealing form of the asymptotic variance

$$\sigma_\alpha^2(f) = (1 + 2\alpha(2\alpha-1)\, D_{2\alpha}(F_0,F))\, \sigma_\alpha^2 + \Gamma^2(\alpha+1)\, \Delta_\alpha(F_0,F) \tag{4.27}$$

follows by taking into account the formula for $D_{2\alpha}(F_0,F)$ obtained from (4.17). The peculiar expressions $\Delta_\alpha(F_0,F)$ figuring in (4.27) can be better understood if we take into account the following facts.

Lemma 4.2. Under the present assumptions about the fixed alternative $F \sim f$, the class $\{\Delta_\alpha(F_0,F) : \alpha \in I\!R - \{0\}\}$ satisfies the relation

$$\begin{aligned} \Delta_\alpha(F_0,F) &= \int_0^1 \left(\frac{f^{-\alpha}}{\alpha} - \int_0^1 \frac{f^{-\alpha}}{\alpha} f\, \mathrm{d}y \right)^2 f\, \mathrm{d}x \\ &= \int_0^1 \left(\frac{f^{-\alpha}}{\alpha} \right)^2 f\, \mathrm{d}x - \left(\int_0^1 \frac{f^{-\alpha}}{\alpha} f\, \mathrm{d}x \right)^2 \end{aligned} \tag{4.28}$$

and this class is continuously extended to all $\alpha \in I\!R$ by putting

$$\begin{aligned}\Delta_0(F_0, F) &= \int_0^1 \left(\ln f - \int_0^1 (\ln f)\, f \,\mathrm{d}y\right)^2 f \,\mathrm{d}x \\ &= \int_0^1 f \ln^2 f \,\mathrm{d}x - \left(\int_0^1 f \ln f \,\mathrm{d}x\right)^2. \qquad (4.29)\end{aligned}$$

All $\Delta_\alpha(F_0, F)$, $\alpha \in I\!R$, are nonnegative measures of divergence of F_0 and F, reflexive in the sense that $\Delta_\alpha(F_0, F) = 0$ if and only if $F = F_0$.

Proof. If $\psi : [0,1] \mapsto I\!R$ is continuous and $F \sim f$ so that

$$\inf_{x\in[0,1]} f(x) > 0 \quad \text{and} \quad \sup_{x\in[0,1]} |\psi(x)\, f(x)| < \infty$$

then

$$\Psi(x) = \int_0^x \psi(y)\, f(y) \,\mathrm{d}y, \quad x \in (0,1)$$

satisfies the equality

$$\int_0^1 (\psi - \Psi/F)^2 f \,\mathrm{d}x = \int_0^1 \psi^2 f \,\mathrm{d}x - \left(\int_0^1 \psi\, f \,\mathrm{d}x\right)^2. \qquad (4.30)$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\Psi^2}{F} = -\left(\frac{\Psi}{F}\right)^2 f + \frac{2\Psi\psi f}{F}$$

so that

$$\begin{aligned}\int_0^1 (\psi - \Psi/F)^2 f \,\mathrm{d}x &= \int_0^1 \psi^2 f \,\mathrm{d}x - \int_0^1 \frac{2\Psi\psi f}{F} \,\mathrm{d}x + \int_0^1 \left(\frac{\Psi}{F}\right)^2 f \,\mathrm{d}x \\ &= \int_0^1 \psi^2 f \,\mathrm{d}x - \left(\frac{\Psi^2(1)}{F(1)} - \lim_{y\downarrow 0} \frac{\Psi^2(y)}{F(y)}\right) \\ &= \int_0^1 \psi^2 f \,\mathrm{d}x - \frac{\Psi^2(1)}{F(1)}\end{aligned}$$

because

$$|\Psi(y)| \le y \sup_{x\in[0,1]} |\psi(x)\, f(x)|$$

and

$$F(y) \ge y \inf_{x\in[0,1]} f(x).$$

Now, by (4.30), (4.28) follows from (4.26). Since f is assumed to be bounded and bounded away from 0,

$$\begin{aligned}\lim_{\alpha\to 0} \Delta_\alpha(F_0, F) &= \int_0^1 \left(\lim_{\alpha\to 0} \frac{f^{-\alpha} - 1}{\alpha} - \int_0^1 \lim_{\alpha\to 0} \frac{f^{-\alpha} - 1}{\alpha} f \,\mathrm{d}y\right)^2 f \,\mathrm{d}x \\ &= \int_0^1 \left(\ln f - \int_0^1 (\ln f)\, f \,\mathrm{d}y\right)^2 f \,\mathrm{d}x \\ &= \Delta_0(F_0, F)\end{aligned}$$

which proves the continuity at $\alpha = 0$. The reflexivity is clear from (4.28) and (4.29). $\square$

If $\alpha > -1/2$ differs from 0 and 1 then the asymptotic variance $\sigma_\alpha^2(f)$ given by (4.27) exceeds the asymptotic variance $\sigma_\alpha^2 = \sigma_\alpha^2(f_0)$ achieved under the hypothesis $F_0 \sim f_0$ by a linear function of σ_α^2 with the coefficients $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ positive unless $F = F_0$. By using Theorem 4.1, we can find the missing formulas for $\sigma_0^2(f)$ and $\sigma_1^2(f)$ by taking limits in (4.27) for $\alpha \to 0$ and $\alpha \to 1$. Since the limits σ_0^2, σ_1^2 were already calculated in Theorem 4.2 and the limits $\Delta_0^2(F_0, F)$, $\Delta_1^2(F_0, F)$ are clear from Lemma 4.2, this last step of the present section is simple, and we can just summarize the results as follows.

Theorem 4.4. The asymptotic formula of (4.23) is valid for all $\alpha > -1/2$ when the alternative $F \sim f$ satisfies the assumptions of Theorem 3.4. The asymptotic means $\mu_\alpha(f)$ are given for all $\alpha \in I\!\!R$ by the explicit formulas (4.14) – (4.17). The asymptotic variances $\sigma_\alpha^2(f)$ are given for all $\alpha \in I\!\!R$ by (4.27) where the explicit formulas for $D_{2\alpha}(F_0, F)$, $\alpha \in I\!\!R$ can be found in (4.16) – (4.17), for σ_α^2, $\alpha \in I\!\!R$ in (4.22) and for $\Delta_\alpha(F_0, F)$, $\alpha \in I\!\!R$ in (4.28) and (4.29). The asymptotic means and variances are continuous in the variable $\alpha \in (-1/2, \infty)$. The asymptotic means satisfy the inequality $\mu_\alpha(f) \geq \mu_\alpha$ mentioned in Theorem 4.2. The asymptotic variances satisfy the inequality $\sigma_\alpha^2(f) \geq \sigma_\alpha^2$. Both equalities take place if and only if $F = F_0$.

Proof. Clear from what was said above. The last inequality and the condition for equality follow from (4.27) where $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ are nonnegative measures of divergence of F_0 and F, equal zero if and only if $F = F_0$. $\square$

Concrete forms of $\mu_\alpha(f)$ and $\sigma_\alpha^2(f_0) = \sigma_\alpha^2$ were illustrated in the Tables 4.1 and 4.2. The next table illustrates $\sigma_\alpha^2(f)$ given by (4.28) for arbitrary f satisfying the assumptions of Theorem 3.4.

Table 4.3 Asymptotic variances $\sigma_\alpha^2(f)$ for selected $\alpha > -1/2$.

α	$\sigma_\alpha^2(f)$		
0	$\sigma_0^2 + \Delta_0(F_0, F)$	$=$	$\frac{\pi^2}{6} - 1 + \int_0^1 f \ln^2 f \mathrm{d}x - \left(\int_0^1 f \ln f \,\mathrm{d}x\right)^2$
$\frac{1}{2}$	$\sigma_{\frac{1}{2}}^2 + \frac{\pi}{4}\Delta_{\frac{1}{2}}(F_0, F)$	$=$	$17 - 4\pi - \pi\left(\int_0^1 \sqrt{f}\,\mathrm{d}x\right)^2$
1	$[1 + \chi^2(F_0, F)]\,\sigma_1^2 + \Delta_1(F_0, F)$	$=$	$\int_0^1 \frac{\mathrm{d}x}{f}\left(\frac{\pi^2}{3} - 2\right) - 1$
$\frac{3}{2}$	$[1 + 6D_3(F_0, F)]\,\sigma_{3/2}^2 + \frac{9\pi}{16}\Delta_{3/2}(F_0, F)$	$=$	$\int_0^1 \frac{\mathrm{d}x}{f^2}\left(\frac{32}{3} - 3\pi\right) - \frac{\pi}{4}\left(\int_0^1 \frac{\mathrm{d}x}{\sqrt{f}}\right)^2$
2	$[1 + 12D_4(F_0, F)\,\sigma_2^2 + 4\Delta_2(F_0, F)]$	$=$	$2\int_0^1 \frac{\mathrm{d}x}{f^3} - \left(\int_0^1 \frac{\mathrm{d}x}{f}\right)^2$
3	$[1 + 30D_6(F_0, F)]\,\sigma_3^2 + 36\Delta_3(F_0, F)$	$=$	$14\int_0^1 \frac{\mathrm{d}x}{f^5} - 4\left(\int_0^1 \frac{\mathrm{d}x}{f^2}\right)^2$

The general form of the asymptotic normality (4.23) established by Theorem 4.4, as well as the continuity of the asymptotic means an variances $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ in the parameter $\alpha > -1/2$ and some explicit formulas for these parameters, seem to be new results. However, in the references cited in Sections 1 and 2 one can find particular versions of these results for some of the statistics U_α from the set $\{T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, S^+_{\phi_\alpha}\}$ on their linear functions, some $\alpha > -1/2$ and some distributions $F \sim f$.

Let us start with the statistic $S^+_{\phi_0}$ proposed by Moran (1951). The asymptotic normality (4.23) for $\alpha = 0$, $U_0 = S^+_{\phi_0}$ and $f = f_0 \equiv 1$ with the parameters $\mu_0(f_0) = \mu_0$ and $\sigma_0^2(f_0) = \sigma_0^2$ given in Tables 4.1 and 4.2 was proved by Darling (1953). The result of Darling was extended to all positively valued step functions f and $\mu_0(f)$ and $\sigma_0^2(f)$ given in Tables 4.1 and 4.3 by Cressie (1976). The result of Cressie was extended to f considered in the present paper and satisfying the Lipschitz condition on $[0, 1]$ by van Es (1992), and to all f considered in the present paper by Shao and Hahn (1995). Cressie and van Es studied $S^+_{\phi_0}$ as the special case obtained for $m = 1$ from a more general statistic based on m-spacings with $m \geq 1$. Van Es used the ideas and methods developed for $m > 1$ by Vasicek (1976) and Dudewicz and van der Meulen (1981).

Greenwood (1946) introduced the statistic

$$\sum_{j=1}^{n+1}(Y_j - Y_{j-1})^2 = \frac{2S^+_{\phi_2} + n + 1}{(n+1)^2}$$

Kimball (1947) proposed the generalization

$$\sum_{j=1}^{n+1}(Y_j - Y_{j-1})^\alpha = \frac{\alpha(\alpha - 1)\,S^+_{\phi_\alpha} + n + 1}{(n+1)^\alpha}, \quad \alpha \in (0, \infty)$$

and Darling (1953) proved an asymptotic normality theorem equivalent to (4.23) for $\alpha \in (0, \infty) - \{1\}$, $U_\alpha = S^+_{\phi_\alpha}$ and $f = f_0 \equiv 1$. Weiss (1957) extended this result of Darling to positive piecewise constant densities f. Hall (1984) obtained the asymptotic normality

$$\frac{1}{\sqrt{n}}\Big(\tilde{U}_\alpha - \alpha(\alpha-1)\,\mu_\alpha(f) - 1\Big) \xrightarrow{\mathcal{D}} N(0, \alpha^2(\alpha-1)^2\sigma_\alpha^2(f)) \quad \text{as } n \to \infty$$

for all statistics

$$\begin{aligned} \tilde{U}_\alpha &= \sum_{j=2}^{n} (n(Y_j - Y_{j-1}))^\alpha \\ &= \alpha(\alpha-1)\,\tilde{S}_{\phi_\alpha} - \alpha n(1 - Y_n + Y_1) + n + \alpha - 1 = \alpha(\alpha-1)\,\tilde{S}_{\phi_\alpha} + n + O_p(1) \end{aligned}$$

(cf. (2.13) for $\phi = \phi_\alpha$ and the proof of Theorem 3.1) with $\alpha \in (-1/2, \infty) - \{0, 1\}$ for any f considered in Theorem 4.4. Here $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ are the same as in Theorem 4.4 and, in fact, this Hall's result was one of the arguments used in the proof of Theorem 4.4.

The statistic $S_{\phi_1}^+$ was proposed recently by Misra and van der Meulen (2001). These authors proved the asymptotic normality (4.23) for $\alpha = 1$, $U_1 = S_{\phi_1}^+$ and arbitrary f considered there, with the parameters $\mu_1(f)$ and $\sigma_1^2(f)$ given in Tables 4.1 and 4.3.

We see that the present Theorem 4.4 unifies and extends the results proved separately in three different situations for two particular statistics from the set (4.5). The formulas for all asymptotic parameters $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ of the statistics U_α are shown to follow via the asymptotic equivalence and continuity in α from Hall's formulas for the asymptotic parameters of $\tilde{U}_\alpha$ with $\alpha \in (-1/2, \infty)$ different from 0 and 1.

References

Anderson, T. W. and Darling, D. A. (1954). A test of goodness of fit. *J. Amer. Statist. Assoc.* **49**, 765–769.

Beirlant, J., Dudewicz, E. J., Györfi, L and van der Meulen, E. C. (1997). Nonparametric entropy estimation: an overview. *Intern. J. Math. Sci.* **6**, 17–39.

Cramér, H. (1928). On the composition of elementary errors. *Skand. Aktuarietids* **11**, 13–74 and 141–180.

Cressie, N. (1976). On the logarithms of high-order spacings. *Biometrika* **63**, 345–355.

Csiszár, I. (1963). Informations theoretische Ungleichung und ihrer Anwendung auf den Beweis der Ergodisität von Markoffschen Ketten. *Publ. Math. Inst. Hungarian Acad. Sci.*, Ser. A, **8**, 85–108.

Darling, D. A. (1953). On a class of problems related to the random division of an interval. *Ann. Math. Statist.* **24**, 239–253.

Darling, D. A. (1957). The Kolmogorov–Smirnov, Cramér–von Mises tests. *Ann. Math. Statist.* **28**, 823–838.

Del Pino, G. E. (1979). On the asymptotic distribution of k-spacings with applications to goodness of fit tests. *Ann. Statist.* **7**, 1058–1065.

Dudewicz, E. J. and van der Meulen, E. C. (1981). Entropy-based tests of uniformity. *J. Amer. Statist. Assoc.* **76**, 967–974.

Durbin, J. (1973). Distribution theory for tests based on the sample distribution function. *Regional Conference Series in Applied Mathematics*, Vol. 9, SIAM, Philadelphia.

Ekström, M. (1999). Strong limit theorems for sums of logarithms of high order spacings. *Statistics* **33**, 153–169.

Freeman, M. F. and Tukey, J. W. (1950). Transformations related to the angular and the square root. *Ann. Math. Statist.* **21**, 607–611.

Greenwood, M. (1946). The statistical study of infections diseases. *J. Roy. Statist. Soc. A*, **109**, 85–110.

Guttorp, P. and Lockart, R. A. (1989). On the asymptotic distributions of high-order spacing statistics. *Canad. J. Statist.* **17**, 419–426.

Györfi, L. and Vajda, I. (2002). Asymptotic distributions for goodness-of-fit statistics in a sequence of multinomial models. *Statistics & Probability Letters* **56**, 57–67.

Hall, P. (1984). Limit theorems for sums of general functions of m-spacings. *Math. Proc. Cambridge Philos. Soc.* **96**, 517–532.

Hall, P. (1986). On powerful distributional tests based on sample spacings. *J. Multivar. Analysis* **19**, 201–224.

Inglot, T., Jurlewicz, T. and Ledwina, T. (1990). Asymptotics for multinomial goodness of fit tests for simple hypothesis. *Theory Probab. Appl.* **35**, 797–803.

Jammalamadaka, S. R. and Tiwari, R. C. (1986). Efficiencies of some disjoint spacings tests relative to χ^2 test. *New Perspectives in Theoretical and Applied Statistics* (Eds. Puri, Vilaplana and Wertz), 311-318. Wiley, New York.

Jammalamadaka, S. R., Zhou, X. and Tiwari, R. C. (1989). Asymptotic efficiency of spacings tests for goodness of fit. *Metrika* **36**, 355-377.

Kimball, B. F. (1947). Some basic theorems for developing tests of fit for the case of nonparametric probability distribution functions. *Ann. Math. Statist.* **18**, 540–548.

Kolmogorov, A. N. (1941). Confidence limits for an unknown distribution function. *Ann. Math. Statist.* **12**, 461–463.

Kuo, M. and Rao, J. S. (1981). Limit theory and efficiencies for tests based on higher order spacings. *Statistics – Applications and New Directions.* Proc. of the golden jubilee conference of the Indian Statistical Institute (Calcutta), 333–352.

Liese, F. and Vajda, I. (1987). *Convex Statistical Distances.* Teubner, Leipzig.

Lindsay, G. G. (1994). Efficiency versus robustness. The case of minimum Hellinger distance and other methods. *Ann. Statist.* **22**, 1081–1114.

Menéndez, M., Morales, D. Pardo, L. and Vajda, I. (1998). Two approaches to grouping of data and related disparity statistics. *Comm. Statist. Theory Meth.* **27**, 609–633.

Misra, N. and van der Meulen, E. C. (2001). A new test of uniformity based on overlapping simple spacings. *Commun. Statist. – Theory Meth.* **30**, 1435–1470.

Morales, D. Pardo, M. C. and Vajda, I.(2003). Limit laws for disparities of spacings. *Nonparametric Statistics* (submitted).

Moran, P. A. P. (1951). The random division of an interval – II. *J. Roy. Statist. Soc. B*, **13**, 147–150.

Morris, C. 1975. Central limit theorems for multinomial sums. *Ann. Statist.* **3**, 165–188.

Neyman, J. (1948). Contribution to the theory of the χ^2 test. *Proc. First Berkeley Symp. on Mathematical Statistics and Probability*, 239–273. Berkeley Univ. Press, Berkeley, CA.

Neyman, J. and Pearson, E. S. (1928). On the use and interpretation of certain test criteria for purposes of statistical inference. *Biometrica* **20A**, 175–247, 264–299.

Pearson, K. (1900). On the criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. *Philos. Mag.* **50**, 157–175.

Pyke, R. (1965). Spacings. *J. Roy. Statist. Soc., Ser. B*, **27**, 395–449.

Pyke, R. (1972). Spacings revisited. *Proc. Sixth Berkeley Symp. Math. Statist. Probability* **1**, 417–427.

Quine, M. P. and Robinson, J. (1985). Efficiencies of chi-square and likelihood ratio goodness-of-fit tests. *Ann. Statist.* **13**, 727–742.

Rao, J. S. and Kuo, M. (1984). Asymptotic results on Greenwood statistics and some of it generalizations. *J. Roy. Statist. Soc., Ser. B*, **46**, 228–237.

Read, T. R. C. and Cressie, N. (1988). *Goodness of Fit Statistics for Discrete Multivariate Data.* Springer, Berlin.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics.* Wiley, New York.

Shao, Y. and Hahn, M. G. (1995). Limit theorems for the logarithm of sample spacings. *Letters* **24**, 121–132.

Slud, E. (1978). Entropy and maximal spacings for random partitions. *Z. Wahrsch. Verw. Gebiete* **41**, 341–352.

Smirnov, N. V. (1944). Approximate laws of distribution of random variables from emprical data. *Uspekhi Matem. Nauk* **10**, 179-206.

Vajda, I. (1989). *Theory of Statistical Inference and Information.* Kluwer, Boston.

Vajda, I. (2003). Asymptotic laws for stochastic disparity statistics. *Tatra Mount. Math. J.* (in print).

Van Es, B. (1992). Estimating functionals related to density by a class of statistics based on spacings. *Scand. J. Statist.* **19**, 61–72.

Vasicek, O. (1976). A test for normality based on sample entropy. *J. Roy. Statist. Soc. B*, **38**, 54-59.

von Mises, R. (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Statist.* **18**, 309–348.

Weiss, L. (1957). The asymptotic power of certain test of fit based on sample spacings. *Ann. Math. Statist.* **28**, 783–786.