Design of Square-Root Derivative-Free Smoothers

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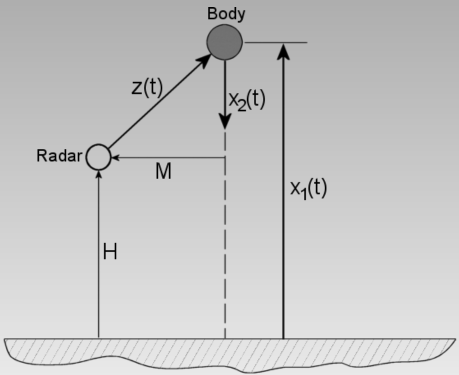
Motivation for State Estimation

What is the state estimation?

The aim of the nonlinear state estimation is to provide the best estimate of unknown (immeasurable, hidden) state of the nonlinear system. This state estimate is based on the a priori information about system and on the available measurable data (maybe noisy or incomplete).

Falling Object Tracking



Known Parameters

- H - radar altitude,
- M - horizontal distance between radar and falling body.

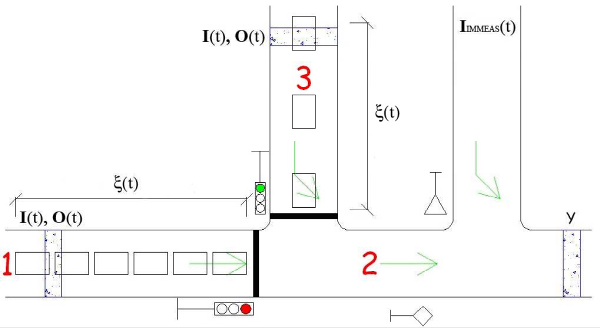
Measurable Variable

- $z(t)$ - t-variant distance between radar and falling body.

Directly Immeasur. Variables

- $x_1(t)$ - t-variant body altitude,
- $x_2(t)$ - t-variant body velocity.

Traffic Model



Measurable Variables

- $\mathbf{O}(t)$ - occupation (proportion of the time period, when the detector is being activated),
- $\mathbf{I}(t)$ - intensity (amount of passing vehicles per time unit).

Directly Immeas. Variable

- $\xi(t)$ - queue lengths on the particular intersection arm.

Description of System

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{w}_k,\ \ k = 0, 1, 2, \ldots$$
$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k,\ \ k = 0, 1, 2, \ldots$$

where

- $\mathbf{f}_k(\cdot)$, $\mathbf{h}_k(\cdot)$ are known vector functions,
- $p_{\mathbf{w}_k}(\mathbf{w}_k) = \mathcal{N}\{\mathbf{w}_k : \mathbf{0}, \mathbf{Q}_k\}$, $p_{\mathbf{v}_k}(\mathbf{v}_k) = \mathcal{N}\{\mathbf{v}_k : \mathbf{0}, \mathbf{R}_k\}$, and $p_{\mathbf{x}_0}(\mathbf{x}_0)$ are known Gaussian probability density functions (pdf's).

State Estimation Problem

The aim of the state estimation is to find the probability density function of the state $\mathbf{x}_k$ conditioned by the measurements $\mathbf{z}^m = [\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_m]$.

$$p(\mathbf{x}_k|\mathbf{z}^m) = ?$$

Solution of the State Estimation Problem

Functional Recursive Relations (FRR's)

- $(k-m)$-step prediction $(k > m)$

$$p(\mathbf{x}_k|\mathbf{z}^m) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{z}^m)d\mathbf{x}_{k-1},$$

- filtering $(m = k)$

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)}{\int p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)d\mathbf{x}_k},$$

- $(m-k)$-step smoothing $(m > k)$

$$p(\mathbf{x}_k|\mathbf{z}^m) = p(\mathbf{x}_k|\mathbf{z}^k)\int \frac{p(\mathbf{x}_{k+1}|\mathbf{z}^m)}{p(\mathbf{x}_{k+1}|\mathbf{z}^k)}p(\mathbf{x}_{k+1}|\mathbf{x}_k)d\mathbf{x}_{k+1}.$$

Exact Solution of the FRR's

The closed-form solution of the FRR's is possible only for a few special cases, e.g. for linear Gaussian system.

Approximative Solution of the FRR's

One of the possible approximative solution of the FRR's is based on such approximation of the system description that the technique of the linear system estimators design can be used in the area of nonlinear systems as well.

Local Estimation Methods

The resultant estimates of the local estimators are given by the mean and the covariance matrix only.

Standard Local Estimation Methods (1970)

The standard local estimation methods are based on the approximation of nonlinear functions in the state or the measurement equation by the Taylor expansion 1st or 2nd order (Lewis, 1986).

Novel (derivative-free) Local Estimation Methods (2000)

The novel local estimation methods are based mainly on the Stirling's polynomial interpolation (Nørgaard et al., 2000) of the nonlinear functions or on the unscented transformation (Julier et al., 2000).

Local Estimation Methods

- Relatively a lot of attention has been paid on the numerical properties of the local, either standard or novel, estimation methods.
- However, the main stress has been laid on the derivation of the numerically stable square-root filters and one-step predictors (Grewal and Andrews, 2001; Nørgaard et al., 2000; van der Merwe and Wan, 2001) and the numerically stable algorithms of the smoothers have not been proposed yet.

Goal of the Paper

- To propose the numerically stable and computationally more efficient square-root smoothing algorithms for the standard and novel estimation techniques.

Types of Smoothing

The smoothing problem $p(\mathbf{x}_k|\mathbf{z}^m)$, $k < m$ can be generally divided into three groups

- *fixed-point smoothing*, when k is fixed,
- *fixed-lag smoothing*, when difference $m - k$ is fixed, and
- *fixed-interval smoothing*, when time instant m is fixed.

Possible Solutions of Smoothing

The smoothing can be solved by two techniques based on

- the extension of the state $\mathbf{x}_m$ by the smoothed state $\mathbf{x}_k$, i.e.

$$\tilde{\mathbf{x}}_m = \begin{bmatrix} \mathbf{x}_m \\ \mathbf{x}_k \end{bmatrix},$$

  and application of common filtering algorithms,
- the solution of the particular FRR to propose a smoothing algorithm.

Algorithm of Rauch-Tung-Striebel Smoother (Lewis, 1986)

One of the possible solution of the smoothing problem $p(\mathbf{x}_k|z^m)$, $k < m$ is given by the Rauch-Tung-Striebel Smoother (RTSS), which is specified by the following relations

$$\hat{\mathbf{x}}_{k|m} = E[\mathbf{x}_k|\mathbf{z}^m] = \hat{\mathbf{x}}_{k|k} + \mathbf{K}_{k|m}(\hat{\mathbf{x}}_{k+1|m} - \hat{\mathbf{x}}_{k+1|k}),$$

$$\mathbf{P}_{k|m} = cov[\mathbf{x}_k|\mathbf{z}^m] = \mathbf{P}_{k|k} - \mathbf{K}_{k|m}(\mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|m})\mathbf{K}_{k|m}^T,$$

where $k = m-1, m-2, \ldots$,

- $\hat{\mathbf{x}}_{k+1|k}$, $\mathbf{P}_{k+1|k}$ are known predictive statistics,
- $\hat{\mathbf{x}}_{k|k}$, $\mathbf{P}_{k|k}$ are known filtering statistics,
- $\mathbf{K}_{k|m} = \mathbf{P}_{xx,k+1|k}(\mathbf{P}_{k+1|k})^{-1}$ is the smoothing gain, and
- $\mathbf{P}_{xx,k+1|k} = E[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})^T|\mathbf{z}^k]$ is state cross-covariance matrix.

Rauch-Tung-Striebel Smoother (cont'd) (Lewis, 1986)

Linearity of the state equation ($\mathbf{f}_k(\mathbf{x}_k) = \mathbf{F}_k\mathbf{x}_k$) allows to find an exact solution of the state cross-covariance matrix in the form

$$\mathbf{P}_{xx,k+1|k} = E[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})^T|\mathbf{z}^k] = \mathbf{P}_{k|k}\mathbf{F}_k^T.$$

Local Rauch-Tung-Striebel Smoothers (Šimandl and Duník, 2006)

- The exact solution of the state cross-covariance matrix cannot be generally found for the system with the nonlinear state equation.
- However, a solution of the state cross-covariance matrix for the nonlinear function $\mathbf{f}_k(\cdot)$ can be found by utilization of the suitable approximation technique e.g. the Taylor expansion, the Stirling's interpolation, the unscented transformation.

Numerically stable version of Rauch-Tung-Striebel type smoothers

The smoothing covariance matrix

$$\mathbf{P}_{k|m} = \mathbf{S}_{k|m}\mathbf{S}_{k|m}^T = \mathbf{P}_{k|k} - \mathbf{K}_{k|m}(\mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|m})\mathbf{K}_{k|m}^T,$$

can be expressed in the square-root form

$$\tilde{\mathbf{S}}_{k|m} = [\mathbf{M}_{k|k} - \mathbf{K}_{k|m}\mathbf{M}_{k+1|k}, \mathbf{K}_{k|m}\mathbf{S}_{Q,k+1}, \mathbf{K}_{k|m}\mathbf{S}_{k+1|m}],$$

where $\mathbf{M}_{k|k}$, $\mathbf{M}_{k+1|k}$, $\mathbf{S}_{Q,k+1}$, and $\mathbf{S}_{k+1|m}$ are known square-root factors of covariance matrices, which form depends on the used approximation techniques.

Unfortunately, matrix $\tilde{\mathbf{S}}_{k|m}$ is rectangular.

Numerically stable version of Rauch-Tung-Striebel type smoothers (cont'd)

To transform the rectangular matrix $\tilde{\mathbf{S}}_{k|m}$ to the square one, the Householder triangularization can be used.

The final form of the square-root smoothing covariance matrix can be expressed as follows

$$\mathbf{S}_{k|m} = ht(\tilde{\mathbf{S}}_{k|m}) =$$
$$= ht\Big([\mathbf{M}_{k|k} - \mathbf{K}_{k|m}\mathbf{M}_{k+1|k}, \mathbf{K}_{k|m}\mathbf{S}_{Q,k+1}, \mathbf{K}_{k|m}\mathbf{S}_{k+1|m}]\Big),$$

where $\mathbf{N} = ht(\mathbf{M})$ stands for Householder triangularization applied to the rectangular matrix $\mathbf{M}$ resulting into the square matrix $\mathbf{N}$ so that the equality $\mathbf{N}\mathbf{N}^T = \mathbf{M}\mathbf{M}^T$ holds.

System Specification (Ito and Xiong, 2000)

$$x_{k+1} = x_k + 5\Delta t x_k(1 - x_k^2) + w_k,$$
$$z_k = \Delta t(x_k - 0.05)^2 + v_k,$$

where

- $k = 1, \ldots, 400,\ \Delta t = 0.01,$
- $w_k \sim \mathcal{N}\{w_k : 0, 0.25\Delta t\},\ v_k \sim \mathcal{N}\{v_k : 0, 0.01\Delta t\},$
- system initial condition is $x_0 = 1.2,$
- and estimators initial condition is $p(x_0|z^{-1}) = \mathcal{N}\{x_0 : 2.2, 2\}.$

The aim is to estimate two-step fixed-lag pdf $p(x_{k-2}|z^k)$.

Experimental Results

Smoother	MSE	Time [s]
Unscented RTSS	1.25×10^{-3}	3.27×10^{-4}
Square-Root Unscented RTSS	1.25×10^{-3}	2.76×10^{-4}

Conclusion Remarks

- The local estimation methods were discussed.
- The application of the Taylor expansion, the Stirling's interpolation, and the Unscented transformation in the design of the local smoothing algorithm was recalled.
- The square-root version of the Rauch-Tung-Striebel type smoother was properly derived.
- The square-root modification of the smoother improves the numerical properties and reduces the computational demands with respect to the "nonsquare-root" algorithms.