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Some Open Problems of Fair Division

The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill.

Einstein, Infeld

Prometheus and Zeus

Hesiodos, in his book Theogonia, mentions how the Greek gods Prometheus and Zeus divided a portion of meat:

Prometheus began by placing the meat into two piles, and Zeus selected one.

Egypt and the British Museum

Some archeological finds had to be divided between **Egypt** and the **Great Britain**. The solution that the leaders of the British and Egyptian expeditions decided upon was to let the British divide everything what had been found between two rooms in the Cairo Museum.

After this was done, a representative from the Egyptian Ministry of Culture studied the objects and claimed the objects in one room, leaving the objects in the other room for the British.

Developed and Developing Countries

The Convention of the Law of the Sea (which went into effect in 1994 with 159 signatories) specifies that

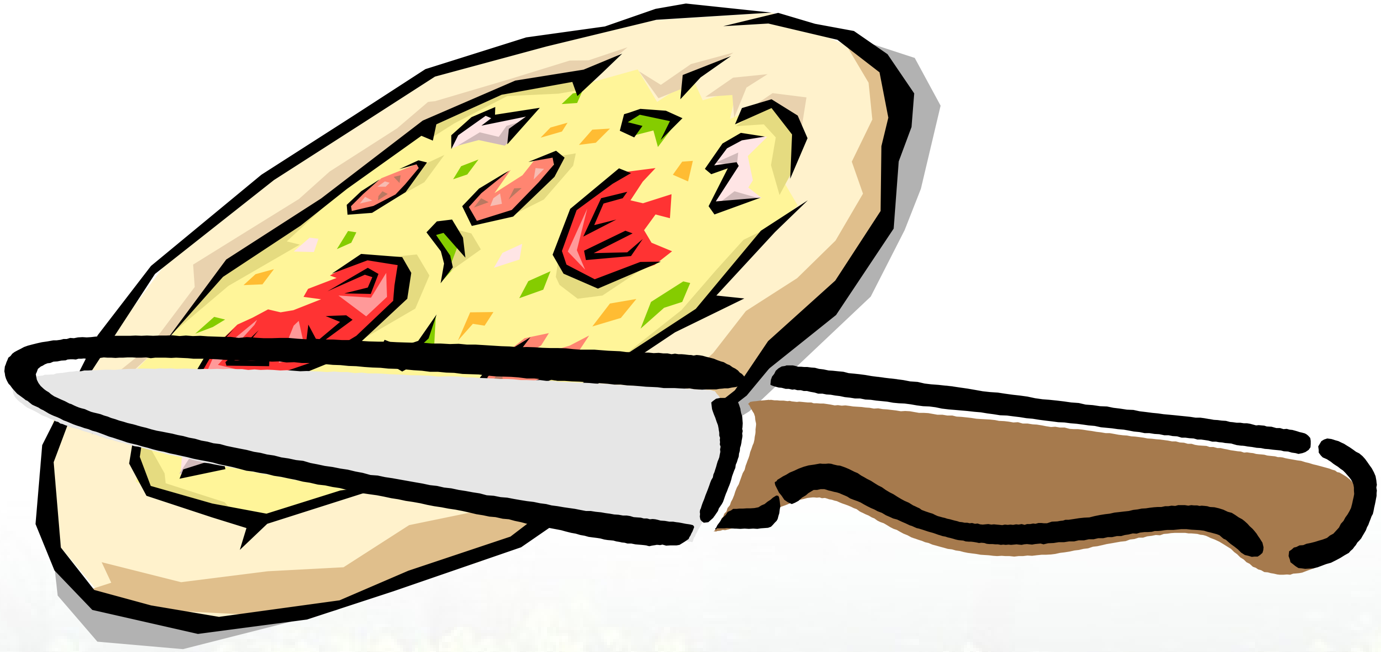
- whenever a **developed country** wants to mine a portion of the seabed, that country must propose a division of the portion in two tracts, and
- then the **Enterprise** (representing the interests of the developing countries through the International Seabed Authority) selects the tracts it prefers, and the developed country receives the remaining tract.

Rules	Strategy of Player 1	Strategy of Player 2
Player 1 divides the cake into two pieces.		
Player 2 chooses a piece.		

Rules	Strategy of Player 1	Strategy of Player 2
Player 1 divides the cake into two pieces.	Divide the cake into two pieces between which you are indifferent.	
Player 2 chooses a piece.		

Rules	Strategy of Player 1	Strategy of Player 2
Player 1 divides the cake into two pieces.	Divide the cake into two pieces between which you are indifferent.	
Player 2 chooses a piece.		Choose the piece you prefer.

Moving knife procedure



First we consider situations where

- the whole cake is a feasible piece,
- the complement of a feasible piece is a feasible piece,
- the union of two feasible pieces is a feasible piece.

This means that the system of feasible pieces is a finitely additive algebra of subsets of the cake.

Solomonic Justice

Two harlots brought a baby, each claiming to be its mother. According to the Bible, Solomon proposed cutting the baby in two, giving half to each woman.

As he sent for a sword, one woman agreed that halves would be a fair division; the other withdrew her claim, and (of course) Solomon gave her the child.

大岡裁き (1677-1751)

江戸時代に大岡越前という名裁判官がいました。あるとき，２人の女性と1人の子供が彼のところに来て，それぞれこの子供は私の子供であると主張しました。困った裁判官は2人の女性に子供の手を引っ張らせ，強いほうが本当の親であると述べました。そこで2人の女性は互いに子供の手を強く引っ張りました。子供は大変泣きましたが，最後に1人の女性が勝ちました。そして，彼女は子供を連れて帰ろうとしました。しかし，裁判官は彼女から子供を引き取り，もう1人の女性に渡し，本当の母親は勝負に負けた方であるといいました。理由はわかるでしょう。

Bah Wang in Han Dynasty (206 BC – 220 AD).

In Yong Chuan, there were two rich brothers. Both of their wives were pregnant simultaneously. Unfortunately, the wife of the elder brother had a miscarriage but she did not tell any one about it. When the wife of the younger brother gave birth to a son, the wife of the elder brother took the baby and claimed that the baby was her son. They kept quarreling for three years about who was the mother.

Finally, they submitted their feud to a judge. The judge, Bah Wang, let one woman size one hand of the baby and the other woman the other hand, and then let the women struggle for the baby. Then, the wife of the younger brother, being afraid of hurting her baby, stopped pulling and let the baby go. Then, Bah Wang said to the wife of the elder brother,

“You claim the baby as your son because you want to get the property of the family. You do not care whether the baby would be hurt or not.”

Regarding preferences, we assume:

- Players’ preferences are represented by complete transitive binary relations on the set of feasible pieces.
- Players’ preferences are monotone with respect to the set inclusion.
- Players have no information about preferences of other players.
- Often the preference relations are given by finitely additive or σ-additive measures.

Assumptions for divide-and-choose

3. Player 1 is able to divide the cake so that he or she is indifferent between these two pieces.
4. No matter how the cake is divided into two pieces, Player 2 will find at least one of the pieces as good as the other one.

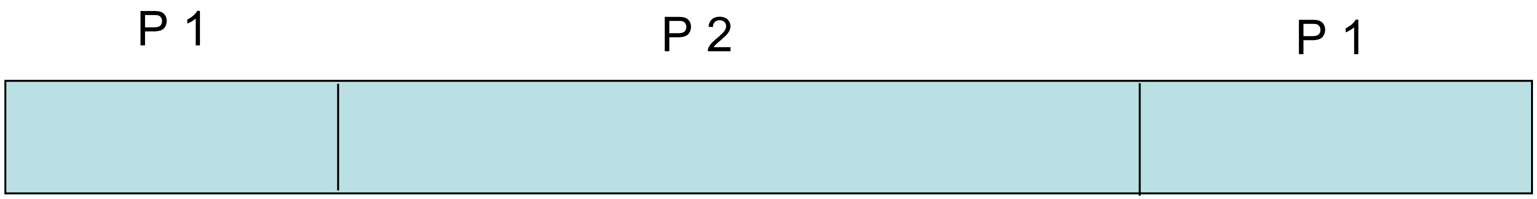
Under these assumptions, there are strategies guaranteeing that:

- Each player gets what he or she perceives to be at least half of the cake.
- No player thinks that the other player received a better piece than he or she did.

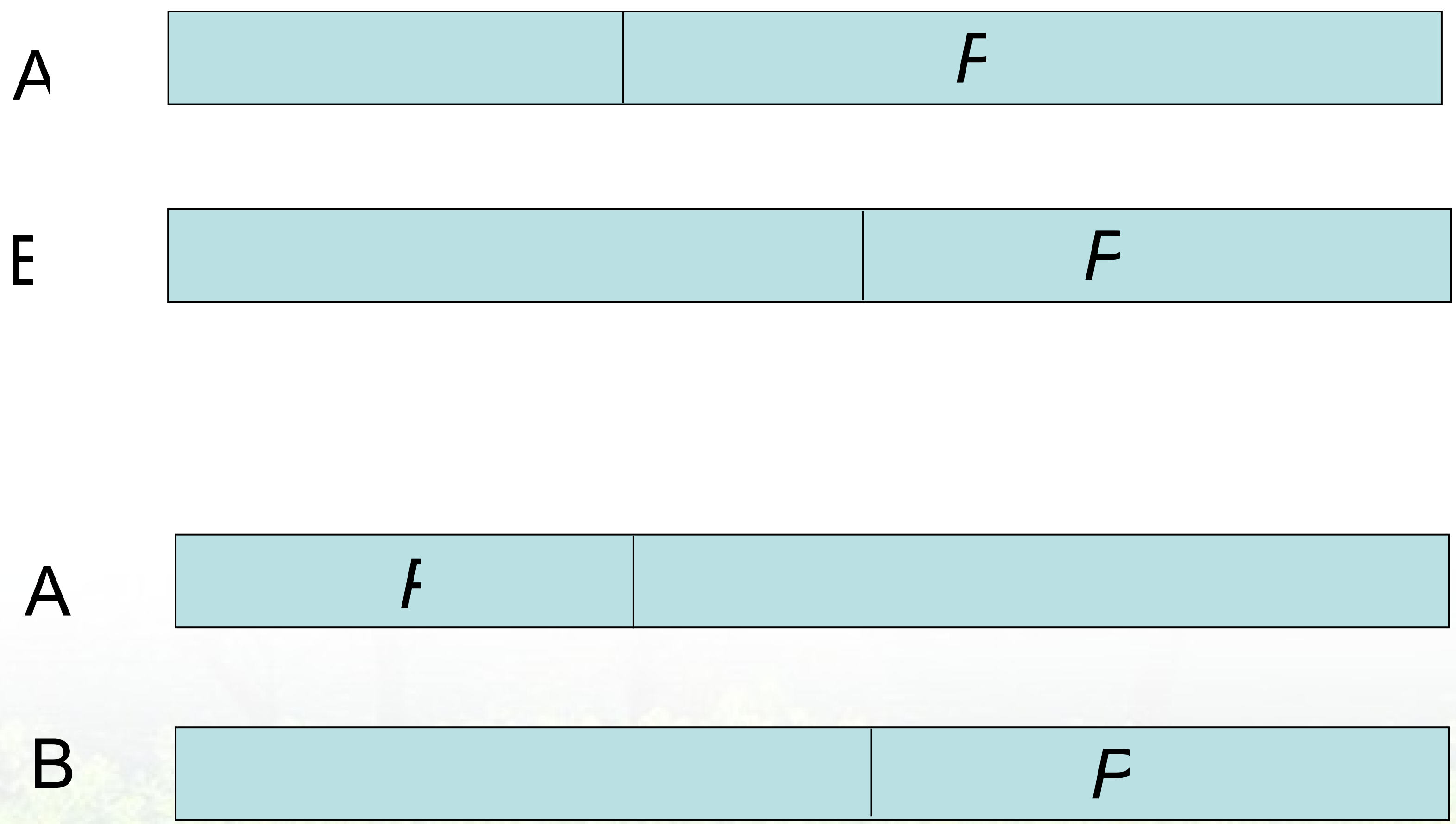
However, there are no strategies guaranteeing that there is no allocation which is better for both players.

Consider the divide-and-choose procedure in the situation where the cake to be divided is ½ vanilla and ½ chocolate. Assume that Player 1 likes only chocolate and Player 2 likes only vanilla. To guarantee himself ½ of the chocolate, Player 1 must divide the cake so that each piece contains an equal portion of chocolate.

Then, regardless of which piece Player 2 chooses, the allocation wherein Player 1 receives all of the chocolate and Player 2 receives all of the vanilla would certainly be better for both players.



Suppose the cake is an interval and the pieces are required to be subintervals.


A
F
E
F
A
F
B
F

Aesop's Fable

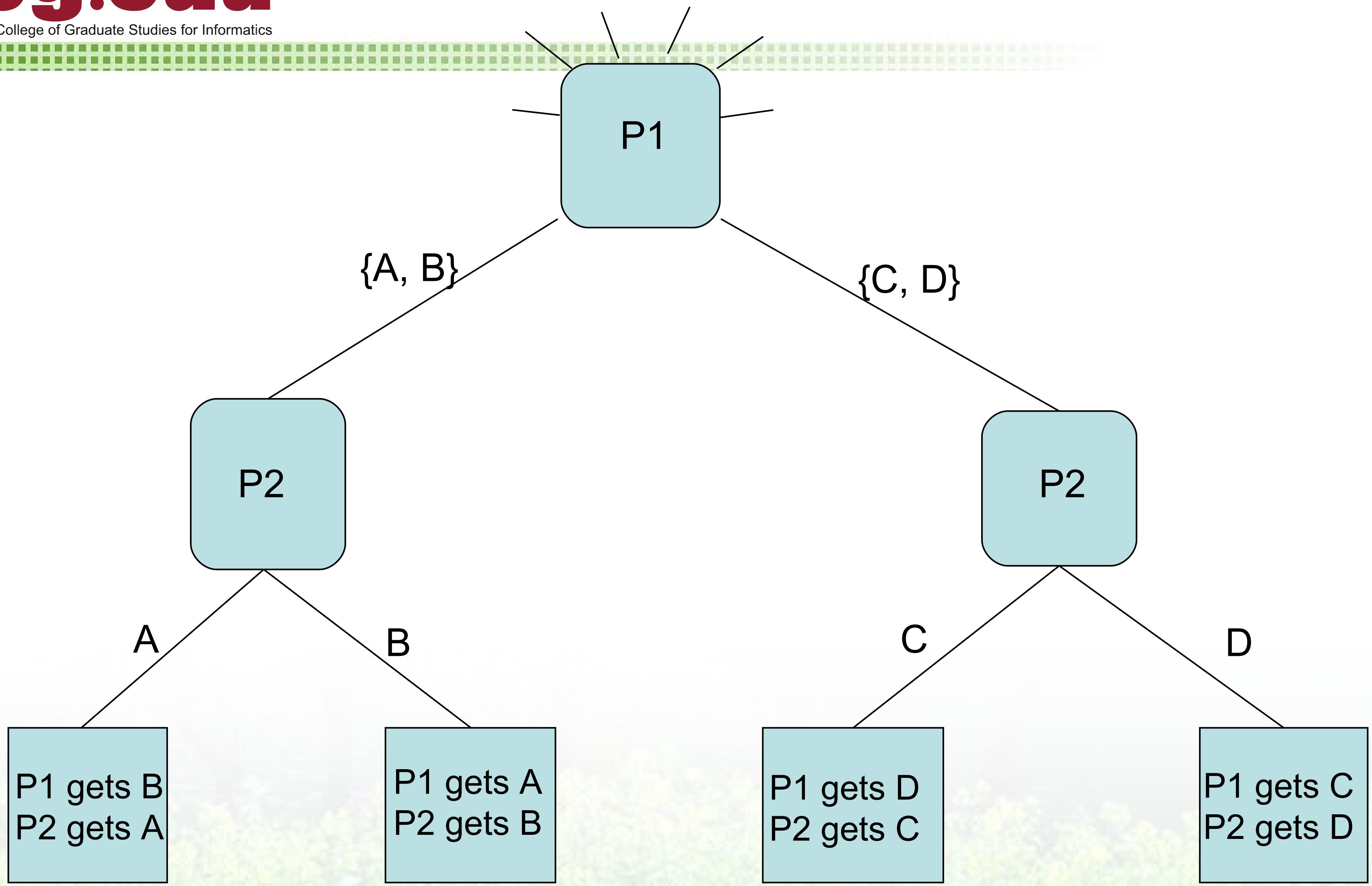
A lion, a fox, and an ass participated in a joint hunt. On request, the ass divided the kill into three equal shares and invited the others to choose. Enraged, the lion ate the ass and then asked the fox to make the division. The fox piled all the kill into one great heap except for one tiny morsel. Delighted at this division, the lion asked:

Who has taught you, my very excellent fellow, the art of division?

to which the fox replied:

I learned it from the ass, by witnessing his fate.

Rules	Strategy of Player 1	Strategy of Player 2
Player 1 divides the cake in two pieces.	Divide the cake in two pieces between which you are indifferent.	
Player 2 chooses a piece.		Choose the piece you prefer.


P1
{A, B}
{C, D}
P2
P2
A
B
C
D
P1 gets B
P2 gets A
P1 gets A
P2 gets B
P1 gets D
P2 gets C
P1 gets C
P2 gets D

The Steinhaus Procedure

Step 1:

Player 1 cuts the cake into three pieces.

Step 2:

Player 2 is given the choice of either passing or to label two of the pieces as bad.

Step 3:

- If Player 2 passed at Step 2, then Player 3 takes any one of the pieces. Player 2 then takes any one of the remaining pieces and Player 1 then receives the remaining piece.
- If Player 2 did not pass at Step 2, then Player 3 is given the same two options that Player 2 had at Step 2. That is, either to pass or to label two of the pieces as bad.

Step 4:

- If Player 3 passed at Step 3, then Player 2 takes any one of the pieces, Player 3 then takes any one of the remaining two pieces, Player 1 then receives the remaining piece.

- If Player 3 did not passed at Step 3, then ???

	S_1	S_2	S_3
Player 1			
Player 2			
Player 3			

	S_1	S_2	S_3
Player 1	○	○	○
Player 2			
Player 3			

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	○	×	○
Player 3			

Player 3 → Player 2 → Player 1

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	○	×	×
Player 3			

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	○	×	×
Player 3	○	×	○

Player 2 → Player 3 → Player 1

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	×	○	×
Player 3	×	○	×

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	×	○	×
Player 3	×	○	×

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	×	○	×
Player 3	○	×	×

	S_1	S_2	S_3
Player 1	○	○	○
Player 2	×	○	×
Player 3	×	○	×

	S_1	S_2	S_3
Player 1	1/3	1/3	1/3
Player 2	2/9	6/9	1/9
Player 3	2/9	5/9	2/9

Step 4:

- If Player 3 passed at Step 3, then Player 2 takes any one of the pieces, Player 3 then takes any one of the remaining two pieces, Player 1 then receives the remaining piece.

- If Player 3 did not passed at Step 3, then Player 1 is given a piece that both Player 2 and Player 3 labeled. The other two pieces are reassembled and the resultant piece re-divided by Player 2 and Player 3.

Assumptions:

- Player 1 is able to divide the cake into three parts so that any one of the parts is acceptable to him or her as fair.
- Given any division of the cake into three parts, each of the remaining players will find at least one of the parts acceptable as fair.
- If a piece is unacceptable both to Player 2 and Player 3, then they can obtain a fair share from dividing the complement.

Under these assumptions:

- There are strategies guaranteeing that each player gets what he or she perceives to be at least one third of the cake.
- There are no strategies guaranteeing that no player thinks that one of the other players received a better piece than he or she did.
- There are no strategies guaranteeing that there is no allocation which is better for all players.

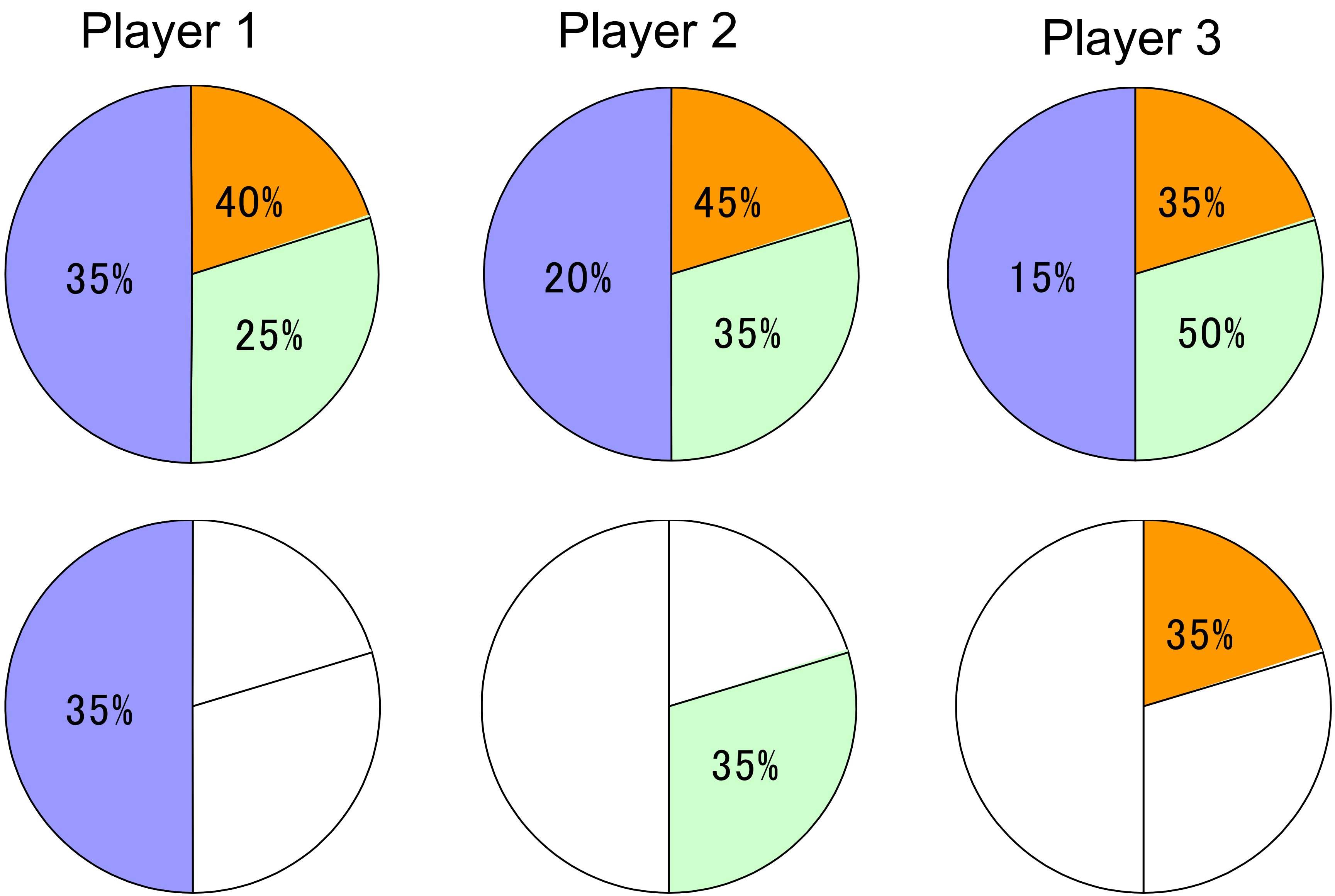
In cut-and-choose, there are strategies guaranteeing that:

- Each player gets what he or she perceives to be at least half of the cake.
- No player thinks that the other player received a better piece than he or she did.

2600 Years Ago

The following solution to the problem of property taxes was inaugurated by the great Athenian statesman, Solon, and used at the tribunal of Athens:

Any citizen who thought that he was paying too high a property tax could exchange his property for that held by anyone who was paying less.

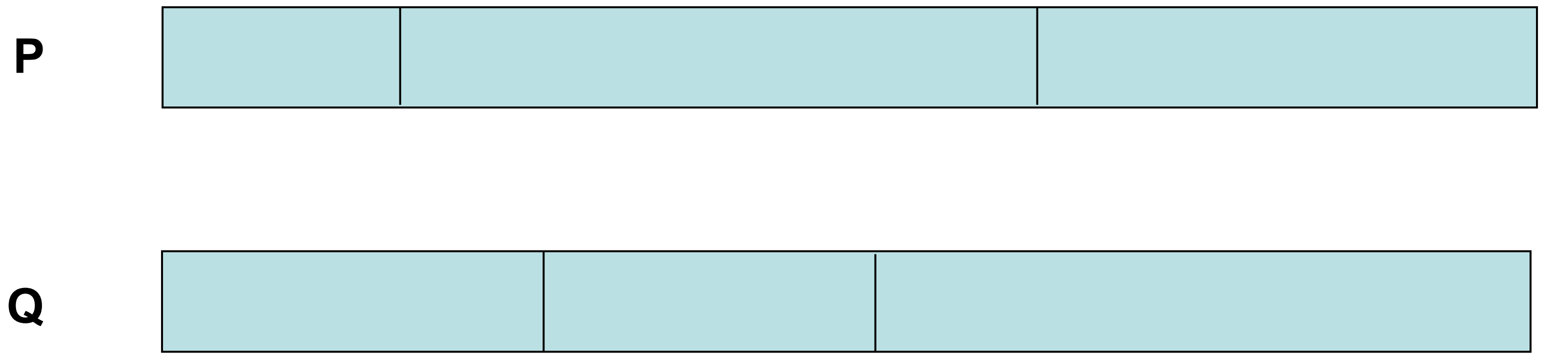

Player 1
Player 2
Player 3
40%
35%
25%
45%
20%
35%
35%
15%
50%
35%
35%
35%

Suppose the cake is an interval and the pieces are required to be subintervals.

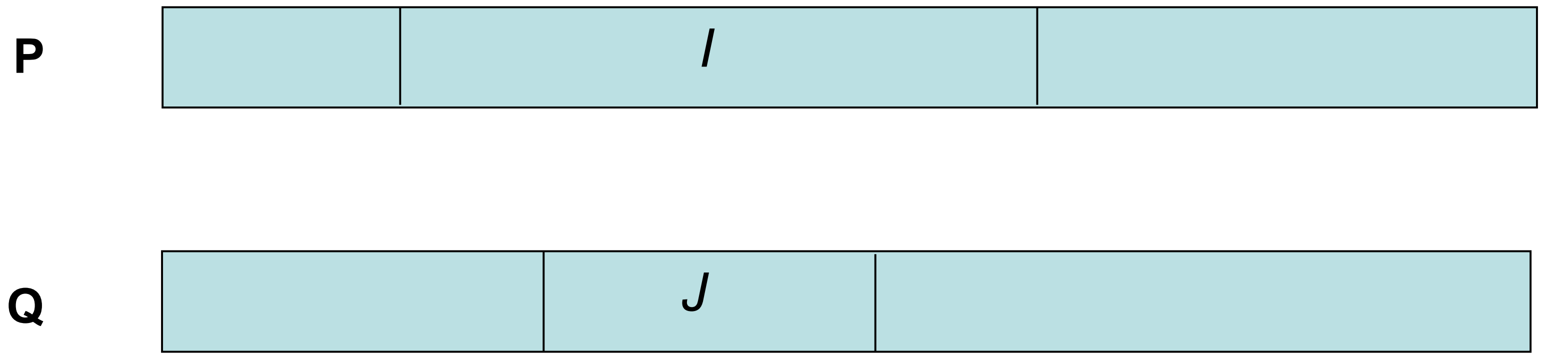
Using a fixed-point theorem, one can show (Stromquist, 1980) that envy-free allocations always exist.

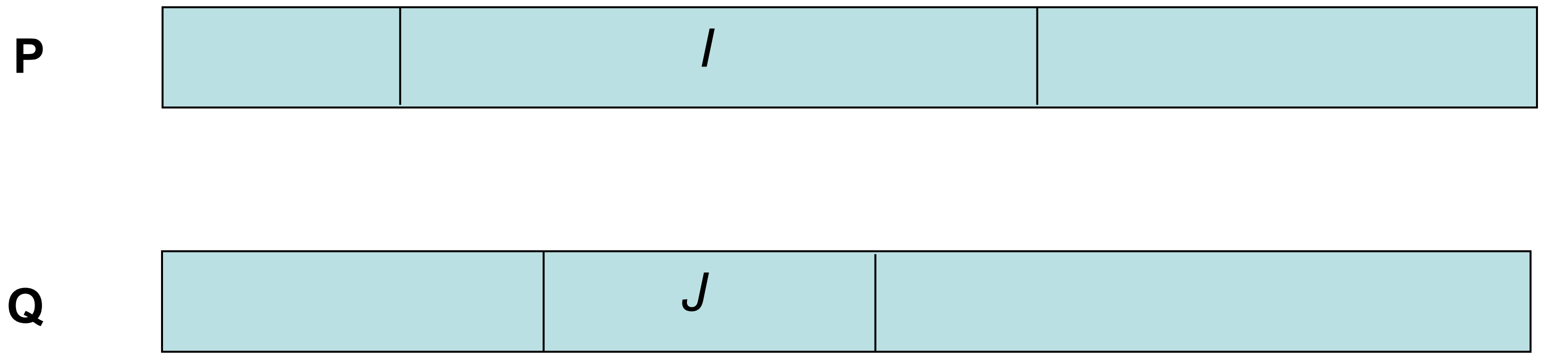
We will say an allocation is *dominated* if there is another allocation which gives all players pieces they strictly prefers.

It turns out that in this “subinterval model” an envy-free allocation is automatically undominated.

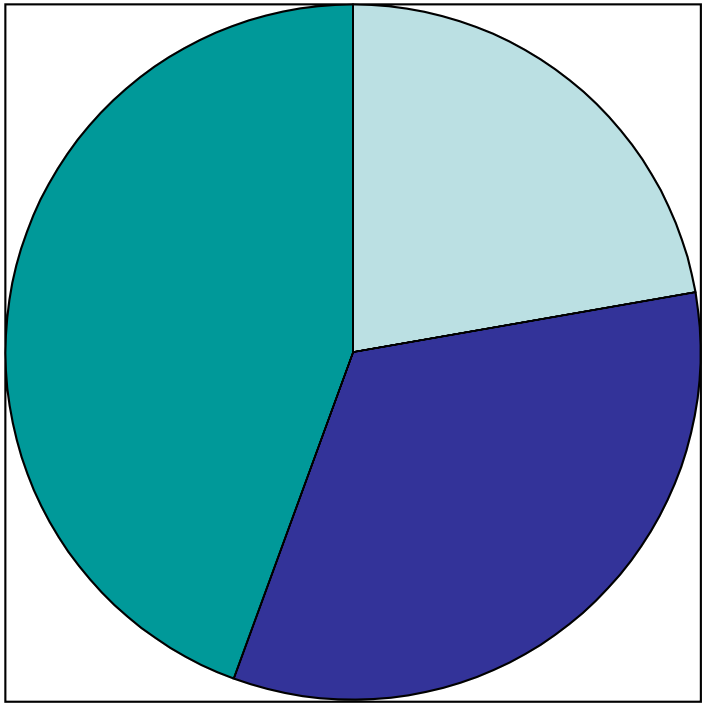


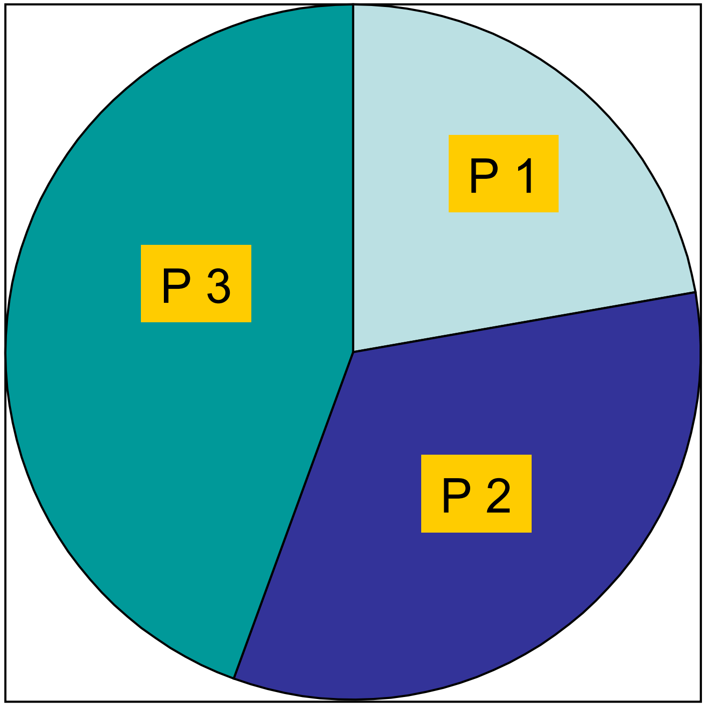
There must be some interval *I* of P which strictly contains some interval *J* of Q.


P
I
Q
J



Someone gets *J* in the allocation **Q**. That player will not be strictly better off then she or he was under **P.**




P 1
P 3
P 2

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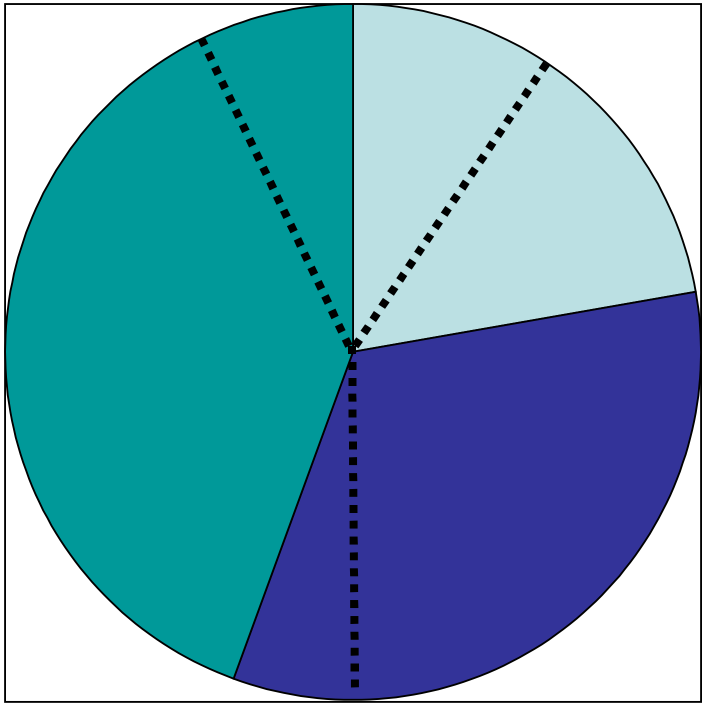

- IS EVERY ENVY-FREE ALLOCATION UNDOMINATED?
- DOES THERE NECESSARILY EXISTS AN ALLOCATION WHICH IS BOTH ENVY-FREE AND UNDOMINATED?

David Gale, 1993

In the case of preferences given by nonatomic measures over a field of feasible pieces, the existence of envy-free divisions for two or more players has been established by means of highly nonconstructive mathematical tools like fixed point theorems, the Borsuk-Ulam theorem, or Lyapunov's theorem.

The Selfridge – Conway Procedure

- Step 1:
  Player 1 cuts the cake into three pieces.

- Step 2:
  Player 2 is given the choice of either passing or trimming a piece from one of the three pieces. The trimmings, if any, are set aside.

- Step 3:

  Player 3 , 2, and 1, in that order, choose a piece from among available pieces observing the following requirement:

  If Player 2 did not pass in Step 2 then he must choose the piece he trimmed, if it was not chosen by Player 3.

- Step 4:

If Player 2 passed at Step 2, we are done. If Player 2 did not pass at Step 2, then either Player 2 or Player 3 chose the untrimmed piece at Step 3.

Let us call this player the "cutter" and the other the "non-cutter". The cutter now divides the trimmings into three pieces.

- Step 5:

The three pieces into which the trimmings are divided are now chosen by the players in the following order:

The non-cutter first, Player 1 second, the cutter third. Then we are done.

Player 1

In step 1, the player should cut the cake into three pieces that he considers to be equally good for him.

In step 5, the player should choose the piece he prefers.

Player 2

In step 2, the player should trim the most preferred piece to create a tie for the best, provided such a tie does not exist after step 1; otherwise he or she should pass.

In step 3, the player should choose the piece he prefers, provided player 3 chose the trimmed piece; otherwise the player has no choice, he or she has to take the trimmed piece.

If the player becomes the cutter in step 4, then the player should cut the trimmings into three pieces that he or she considers to be equally good; otherwise the player should choose the best piece in step 5.

Player 3

In step 3, the player should choose the best piece.

In step 4, if the player becomes the cutter, then the player should cut the trimmings into three pieces that he or she considers to be equally good; otherwise the player should choose the best piece in step 5.

To verify that such strategies produce an envy-free division, we observe that, after the third step, either the whole cake or its part (without trimmings) is divided in envy-free way.

Player 3 envies no one because he or she chooses first.

Player 2 can secure the best piece for himself or herself by trimming.

Player 1 does not receive the trimmed piece.

It remains to examine the division of trimmings.

- The non-cutter envies no one because he is choosing first.
- The cutter is non-envious because he created equal pieces.
- Player 1 does not envy the cutter because he chooses before the cutter does, and he does not envy the non-cutter because the non-cutter gets the trimmed piece together with some part of the trimmings, which is certainly no better than the untrimmed piece of Player 1.

A Variation on Aesop's Fable

Four animals find a treasure and must decide how to divide it fairly. The lion speaks up and says:

"First we must carefully divide the treasure into four parts. The first part goes to me, since I am king of beasts. The second part is mine, owing to my strength. The third part is mine because of my courage. As for the fourth part, anyone who cares to dispute it with me can do so, at his own risk."

Brams and Taylor (1995) showed how to construct, in a finite number of steps, an envy-free division for an arbitrary finite number of players under the following two assumptions.

- For every piece *P* and every positive integer *k,* piece *P* can be partitioned into *k* pieces of equal measure.
- For every pieces *P* and *Q,* either *P* can be trimmed to yield a subset the same measure as *Q*, or vice-versa.

The rules of the Selfridge-Conway procedure have the following nice property.

The number of necessary steps and cuts in the worst case (1 cut for the cut-and-choose, 5 cuts for the Selfridge-Conway) does not depend on players' preferences.

However, the number of necessary steps and cuts in the Brams-Taylor procedure depends on players' preferences and can be arbitrarily great even in the case of four players.

Is there a game-theoretic procedure for four (or more) players such that

- it has a finite horizon;
- the number of necessary steps and cuts can be bounded by a number that is independent of players' preferences;
- it is envy-free in the sense that each player can follow the rules in such a manner that he or she is non-envious in the resulting division, regardless of which strategies the other players follow.

Be fair if you can

Thank you

S. J. Brams and A. D. Taylor. *An envy-free cake division protocol.* Amer. Math. Monthly 102 (1995), 9-18.

S. J. Brams and A. D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution.* Cambridge University Press, Cambridge, UK, 1996.

L. E. Dubins and E. H. Spanier. *How to cut a cake fairly.* Amer. Math. Monthly 68 (1961), 1-17.

D. Gale. *Mathematical Entertainments.* Mathematical Intelligencer 15 (1993), 48-52.

H. Steinhaus. *The Problem of Fair Division.* Econometrica 16 (1948), 101-104.

W. Stromquist. *How to cut a cake fairly.* Amer. Math. Monthly 87 (1980), 640-644.

Everything has been thought of before, but the problem is to think of it again.

Goethe

Unofficial Answers

- Who cares.
- Whatever the origin of the problem might be, the problem is interesting and challenging.
- The problem was created as a by-product of our previous research.

Official Answers

- The problem under consideration appears almost everywhere in Nature.
- Our research represents an important issue associated with many real world problems.
- Many interesting instances of the problem can be handled by our approach.