Equilibrium Behaviour of Zero Range Processes on Binary Tree

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Abstract

We focus on describing an interacting particle system in the case when the set of sites, on which the particles move, has a structure different from the usually considered set $\mathbb{Z}^d$. We have chosen the tree structure with the dynamics that leads to one of the classical particle systems, called the zero range process. The reason for this choice was given by a motivation from queueing systems and networks, since the zero range process corresponds to an infinite system of queues and the arrangement of servers in the tree structure is natural in a number of situations.

The main result of this work is a characterisation of invariant measures for some important cases of site-disordered zero range processes on a binary tree. Namely, the case when the single particle law is a simple random walk on a binary tree. Another result is connected with the speed of convergence to equilibrium for the latter zero range process.

Introduction

Let us briefly introduce the so-called zero range process which is one of the interacting particle systems presented in [4]. We start with a description of a general interacting particle system. Let us have an arbitrarily large system of indistinguishable particles (customers or generally units) which move among sites (queues or generally nodes) in separate jumps. Let us denote by X the set of the sites. By a jump we mean that one particle leaves its site for one of its neighbouring sites. To describe the dynamics of jumps, we can imagine that there is an exponential clock at every site, all the clocks are mutually independent, and when the clock at site x rings one particle jumps from x to y with a probability $p(x,y)$. "Exponential clock" means that epochs between rings are independent exponentially distributed random variables. The rate of the jump then depends at least on the existence of a particle at the leaving site x and in general can even depend on the number of particles at all sites. This is the reason why we call these particle systems interacting ones. Let us denote by $\eta(x)$ the number of particles at site x and by $\eta = (\eta(x) : x \in \mathrm{X})$ one particular configuration of the whole particle system. Moreover, we suppose that each site $x \in \mathrm{X}$ has its own characteristic $\lambda_x > 0$, which we call the leaving rate.

Specifying the jump rates, one can obtain a whole range of interacting particle systems which differ by the type of their interactions. We are interested in the particular types of interactions called "zero range," for which the jump rate is only a function $g(\eta(x))$ of the number of particles at the leaving site and its leaving rate λ_x. It implies that interactions can occur among particles at the same site. A particle system with these interactions is then called a **zero range process**.

Set X of the sites on which the particles move can be finite or countable, typically $\mathbb{Z}^d$ or a d-dimensional discrete torus. We shall investigate in this paper the case where X is a binary tree. We assume that the only possible movement of particles is between sites which are neighbours on the tree. Each node $x \in \mathrm{X}$, except for the root, which has no ancestor, has exactly three neighbours: its *ancestor* (parent) x^-, its *left descendant* (child) x^+ and its *right descendant* (child) x_+. We shall use notation $x{\sim}y$ for "x and y are neighbours".

Since the evolution of a system is given by individual jumps between sites, a configuration of particles η can be changed if one particle jumps from a site $x \in \mathrm{X}$ to another site $y \in \mathrm{X}$. We denote this changed configuration by

$$\eta^{xy}(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x \\ \eta(y) + 1 & \text{if } z = y \\ \eta(z) & \text{otherwise.} \end{cases}$$

The transition $\eta \longmapsto \eta^{xy}$ for some $x \neq y$ is the only possible transition in one jump.

We consider that this system evolves in time and its description is given by the number $\eta_t(x)$ of particles at each site x at each time t. Since its dynamics is random and follows from exponentially distributed clocks, we are going to define the particle system as a Markov process with continuous time. We denote by $\theta(\eta, \zeta)$ the **transition rates** for every $\eta \neq \zeta$. According to the previous description we put

$$\begin{aligned} \theta(\eta, \zeta) &= \theta(\eta, \eta^{xy}) && \text{if } \zeta = \eta^{xy} \text{ for any } x \neq y \\ &= 0 && \text{otherwise,} \end{aligned}$$

for every $\eta \neq \zeta$ where $\theta(\eta, \eta^{xy}) = g(\eta(x))\lambda_x p(x,y)$ for every $x \neq y$.

If we specially consider $|\mathrm{X}| < \infty$ then the zero range process on X is a continuous-time Markov process with generator matrix $Q = \big(\theta(\eta,\zeta)\big)_{\eta,\zeta\in\mathfrak{X}}$. In the case when X is infinite we define zero range processes in the following way.

Definition 1

Let X be a full binary tree. Furthermore, let us consider

- *bounded function $g : \mathbb{N} \to [0,\infty)$, such that $g(0) = 0$, $g(k) > 0$ otherwise, called* **the speed function**
- *transition probability $(p(x,y) : x, y \in \mathrm{X})$, called* **the single particle law**, *such that $p(x,y) = 0$ if $x \not\sim y$,*
- *constants λ_x, such that $0 < \lambda_x \leq \Lambda$ for every $x \in \mathrm{X}$, called the leaving rates or* **the environment** *when we mean the whole family $(\lambda_x : x \in \mathrm{X})$.*

Then **the zero range (ZR) process** *on binary tree X with speed function g, with single particle law $p(x,y)$ in non-homogeneous environment $(\lambda_x : x \in \mathrm{X})$, is the canonical Markov process $(P^\eta : \eta \in \mathfrak{X})$ with state space*

$$\mathfrak{X} := \mathbb{N}^{\mathrm{X}} = \{\eta : \mathrm{X} \to \mathbb{N}\}$$

given by the infinitesimal generator

$$(\mathcal{L}f)(\eta) = \sum_{x\in\mathrm{X}} \sum_{y\sim x} g(\eta(x))\ \lambda_x\ p(x,y)\ [f(\eta^{xy}) - f(\eta)] \qquad (1)$$

for every $\eta \in \mathfrak{X}$ and cylinder function f on $\mathfrak{X}$.

A function $f : \mathbb{N}^{\mathrm{X}} \to \mathbb{R}$ is called **the cylinder function** if there exists a finite $K \subset \mathrm{X}$ such that $f(\eta) = f(\zeta)$ holds $\forall\ \eta, \zeta \in \mathrm{X} : (\eta(x) = \zeta(x)\ \forall\ x \in K)$.

Invariant measures

A very interesting problem concerning particle systems is to describe the set of invariant measures. A good reference on these problems is paper [1]. In this paragraph we introduce some family of probability measures on $\mathfrak{X}$ which is a typical family of invariant measures with respect to the zero range processes.

A probability measure μ on $\mathfrak{X}$ is called **invariant** *with respect to the zero range process given by generator (1) if*

$$\int \mathcal{L}f\ d\mu = 0 \quad \text{for each cylinder function } f \text{ on } \mathfrak{X}.$$

The following result is not surprising and follows from the specific tree structure. For more details see [2].

Theorem 2 *Let us assume $g(k) = \mathrm{I}_{[k>0]}$ and $p(x,y) \neq 0$ iff $x \sim y$. Then product measures ν^φ defined on space $\mathbb{N}^{\mathrm{X}}$ by its marginal distributions:*

$$\nu^\varphi(\eta : \eta(x) = k) = \Big(\frac{\varphi\ \ pop(x)}{\lambda_x popr(x)}\Big)^k \Big(1 - \frac{\varphi\ \ pop(x)}{\lambda_x popr(x)}\Big) \quad \forall k \in \mathbb{N}.$$

$\forall x \in \mathrm{X}$, is invariant for the zero range process for tree X. Here φ is an arbitrary nonnegative constant satisfying $\varphi < \lambda_x \frac{popr(x)}{pop(x)}\ \forall x{\in}\mathrm{X}$.

Notation: $pop(x)$ is an abbreviation for the probability of "path from the root to node x at the k-th level" and similarly $popr(x)$ abbreviates the probability of "path from x back to the root".

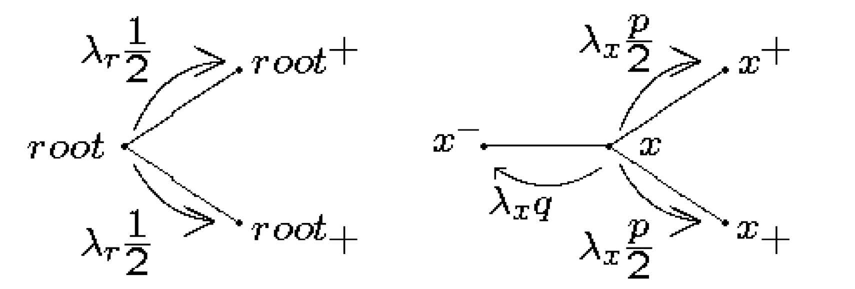
Remark: Let us denote

$$c := \inf_x \lambda_x \frac{popr(x)}{pop(x)} > 0.$$

If $c > 0$ then Theorem 2 gives a whole set of invariant measures for a given ZR process which is indexed by parameter φ. We get
$\varphi \in [0, c)$ when c is attained by some $x \in \mathrm{X}$ and
$\varphi \in [0, c]$ when c is not attained.

Simple random walk on tree

We focus now on a very interesting special case - we consider that the single particle law of our ZR process is a *simple random walk on the tree*. We mean that the jump probabilities at each node (except for the root) are the same: a particle can jump from a given site to its ancestor with probability q, to its left descendant with probability $p/2$ and to its right descendant with probability $p/2$ too, where $p + q = 1$ and $0 < p, q < 1$. The probability of a jump from root to the left is the same as to the right, equal to $1/2$. The rates λ_x are considered to be arbitrary laying in an interval $(0, \Lambda]$.



Theorem 2 gives specially in this case the following set of product invariant measures:

$$\nu^\varphi(\eta : \eta(x) = k) = \Big(\frac{\varphi}{\lambda_x}(\frac{p}{2q})^{|x|}\Big)^k \Big(1 - \frac{\varphi}{\lambda_x}(\frac{p}{2q})^{|x|}\Big) \qquad (2)$$

$\forall x \in \mathrm{X}$, $\forall k \in \mathbb{N}$, where $\varphi \in \Phi_\lambda$ and

$$\Phi_\lambda := \{0 \leq \varphi < \lambda_x\Big(\frac{2q}{p}\Big)^{|x|}\ \forall x \in \mathrm{X}\}.$$

There is a natural classification by parameter p, constant c and the expected total number of particles on the binary tree

$$RT_\varphi := \sum_{n=0}^{\infty} (\frac{p}{2q})^n \sum_{|x|=n} \frac{\frac{\varphi}{\lambda_x}}{1 - \frac{\varphi}{\lambda_x}(\frac{p}{2q})^{|x|}}$$

(i) $RT_\varphi < \infty$ for some $\varphi \in (0, c)$
(ii) $p \leq q$ & $RT_\varphi = \infty$ for some $\varphi \in (0, c)$ & $c \neq 0$
(iii) $p > q$ & $p/2 \leq q$
(iv) $c = 0$, excepting case (iii).

Results concerning characterization of the set $\mathcal{I}$ of invariant measures.

(i) The measures $\nu_K = \nu^\varphi(\cdot \mid \sum_{x\in\mathrm{X}} \eta(x) = K)$ on $\mathbb{N}^{\mathrm{X}}$ carried on configurations with exactly a finite number K of particles are invariant measures which even form set $\mathcal{I}_e$ of extreme points for the set of all invariant measures. If we start with an infinite configuration then the clustering of particles will occur at nodes z such that $\lambda_z(\frac{2q}{p})^{|z|} = c$.

(ii) In this case the characterisation of $\mathcal{I}$ is the following:

$$\mathbf{V} := \{\nu^\varphi : \varphi \in \Phi_\lambda\} = \mathcal{I}_e.$$

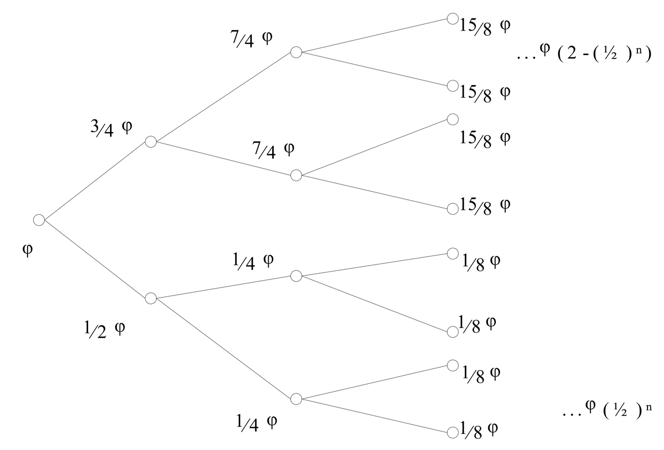
(iii) This case is abundant in the number of invariant measures and set $\mathbf{V}$ of invariant measures following from Theorem 2 is too small to describe the whole set $\mathcal{I}$ of invariant measures in this case. See example below.

(iv) The measure ν^0 (which is the Dirac measure carried on the zero configuration) is the only invariant measure for this zero range process.

Example 3 *Let us consider $p = 2/3$, i.e. a special case of (iii). The product measure $\nu^{\pi_{\varphi,\lambda}}$ with marginals*

$$\nu^{\pi_{\varphi,\lambda}}(\eta(x) = k) = (\frac{\pi_\varphi(x)}{\lambda_x})^k \Big(1 - \frac{\pi_\varphi(x)}{\lambda_x}\Big) \qquad \forall k \geq 0\ \forall x \in \mathrm{X}$$

with $\pi_\varphi(x)\ \forall x$ defined by the picture:



is invariant for the zero range process.

For more details see [2].

Speed of convergence to equilibrium

Let us consider the model from previous paragraph, where the single particle law is a simple random walk on the full rooted binary tree. We consider $p \leq 2/3$ (we abandon not much interesting case (iv)). However let us, in addition, assume that environment $(\lambda_x : x \in \mathrm{X})$, which was till now arbitrary, is chosen as:

$$\lambda_x = (\frac{p}{2q})^{|x|} \text{ for every } x \in \mathrm{X}. \qquad (3)$$

The reason is that then invariant measures (2) are site independent and each its marginal distribution is the same geometrical distribution with parameter φ.

A known result called **Equivalence of ensembles** gives in this case the following statement: *For every $\rho \in [0, \infty)$*

$$\mu_{n,\lfloor\rho|\mathrm{X}_n|\rfloor}\Big(\eta_n \in \mathfrak{X}_{n,\lfloor\rho|\mathrm{X}_n|\rfloor} : \eta_n(x_i){=}k_i,\ 1 \leq i \leq l\Big) \overset{n\to\infty}{\longrightarrow} \nu^{\Phi(\rho)}\Big(\eta \in \mathfrak{X} : \eta(x_i) = k_i,\ 1 \leq i \leq l\Big)$$

pointwise, where for every $n, K > 0$ measure $\mu_{n,K}$ is the uniform distribution on state space of finite approximations:

$$\mathfrak{X}_{n,K} := \{\eta \in \mathbb{N}^{\mathrm{X}_n} : \sum_{x\in\mathrm{X}_n} \eta(x) = K\}.$$

$\mathrm{X}_n \subset \mathrm{X}_{n+1}$ are finite connected subset of infinite tree X, $\bigcup \mathrm{X}_n = \mathrm{X}$, K is a fixed number of particles on finite set of sites X_n. Here $\Phi(\rho) = \frac{\rho}{\rho+1}$.

It is often useful approach to consider in this way finite approximation of our zero range process. We need only suitably restrict the single particle law $p(x,y)$ on X_n. Let us call them (n,K)-finite ZR processes. Notice that they are finite state space Markov processes with the unique invariant measure, the uniform distribution $\mu_{n,K}$ and, moreover, under assumption (3) each (n,K)-finite ZR process is reversible w.r.t. equilibrium measure $\mu_{n,K}$.

Now we bring out a result concerning the speed of convergence of each (n,K)-finite zero range process to its equilibrium. If we denote by $P_t^{n,K}$ the associated transition probability matrix at time t then we obtain the following statement.

Theorem 4 *There exists a constant $0 < C < \infty$ such that for every $n \in \mathbb{N}$, $K > 0$, f on $\mathfrak{X}_{n,K}$, $t > 0$:*

$$||P_t^{n,K} f - \mathrm{E}_{\mu_{n,K}} f||_{L_2(\mu_{n,K})} \leq \|f\|_{L_2(\mu_{n,K})} \exp\Big\{\frac{-t}{Cn2^n(\frac{2q}{p})^n\Big(1 + \frac{K}{2^n}\Big)^2}\Big\}.$$

This result is a consequence of a rather general result on Poincaré inequality from [3] and an employing the graph structure of the binary tree. Cf. [2].

Acknowledgements

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References

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