Prescriptive Inductive Operations On Probabilities Serving to Decision-Making Agents

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Abstract—**Approximation, extension and merging of probability distributions support inductive reasoning. They serve to modelling, knowledge and preference elicitation as well as to a soft cooperation within various decision-making (DM) scenarios. The theory dubbed as fully probabilistic design of DM strategies unifies the design of these operations on distributions. The unification decreases a danger of their improper choice and use. Still there is an uncertainty how the gained tools should be wielded. The paper diminishes it by spelling out conditions ruling their exploitation. The paper serves as an updated description of these tools, provides examples of their use and guides their tailoring to diverse scenarios.**

Keyword dynamic decision making, uncertainty, relative entropy (RE), approximation, merging, extension

I. Introduction

The fully probabilistic design of decision strategies (*FPD*, [1], [2], [3], [4]) is a prescriptive theory of dynamic DM under uncertainty. It serves us as a unifying DM framework for solving the problems addressed in this paper.

DM is broadly understood as a targeted choice among given options, [5], [6]. It aims to influence behaviour of the closed loop formed by the DM agent and its environment. The behaviour means the collection of all observed, opted and thought of random variables within the closed loop. Their joint probability density (*pd*) models the behaviour. FPD non-traditionally uses an opted ideal joint pd for specifying the desired behaviours. The ideal pd quantifies DM aims by assigning high values to desired behaviours and small to undesired ones. With these joint pds, FPD deals with their, *deductively* derived, marginal and conditional pds (called *slice pds*) modelling behaviour slices. They serve to the knowledge accumulation (learning) and the optimal-strategy design.

A. Paper Topic and Aims

The paper cares about agents that intend to exploit a prescriptive DM theory. Any such agent needs quantitative *DM elements* of the employed DM theory for solving learning and strategy-design problems. A mapping of real-life induced DM knowledge and aims on them is a collection of hard problems addressed by domain-specific modelling, knowledge [7], [8] and preference [9], [10] elicitation as well as by designs of quantitative cooperation and negotiation tools [11], [12], [13]. In spite of permanent progress, the mapping is far from being fully matured and sufficiently unified. It is extremely costly both in design and application phases. The current paper contributes to overcoming this state within FPD, which covers standard DM under uncertainty [4], has been non-trivially used [14], [15] and quantifies beliefs and preferences in a unified probabilistic way [2] suitable for cooperation [13], [16], [17].

The pair of joint pds forms DM elements of the adopted FPD. A real agent quantifies them only partially. Their inevitable completion is a DM task on its own. The completion must be based on *inductive reasoning* due to the generic non-uniqueness [18], even non-existence [19], of a joint pd compatible with the practically accessible information. Obviously, the adopted inductive reasoning strongly influences the final DM quality. This calls for *deductive solutions of common inductive tasks*. Their construction and use form the central topic of the paper. The inspected approximation, extension and merging[1] of pds are such basic tasks. All of them are DM tasks as they opt among possible inductive reasoning ways. These *DM meta-tasks* have often been addressed, see comments within the text and Sec. IV. The work [20] is the most advanced one in the FPD framework. Even within it, solutions differ [13] and their assumptions meagerly guide how to opt among them. The paper *aims* to: (i) correct this state; (ii) improve past solutions of DM meta-tasks; and (iii) add practical, insight-offering, examples of their use.

B. Layout

The rest of this section provides notation, notions and basic relations. Sec. II builds an approximator of a given pd as a minimiser of the DM-relevant proximity measure of pds. It gives an example of its use and discusses its (non)ambiguity. Sec. III similarly deals with extensors of a partial information. The examples indicate their wide usability in modelling, merging, cooperative learning and soft cooperative DM. The width is unsurprising as any inductive reasoning essentially extends a partial information. The text refers to the related works gradually and in Sec. IV, where extra comments on predecessors [13], [20] exploit knowledge of the technical paper's content. Sec. V lists achievements and open problems.

Propositions can be characterised as follows: Prop. 1 reflects methodological basis and other propositions are its applications. Its axiomatic basis is in [3], [4]. Approximation principle, given by Prop. 2, guides how to approximate a given pd by a pd from a given set. Prop. 3 shows that the popular moment fitting is an application of Prop. 2. Minimum relative-entropy principle, described by Prop. 4, serves as universal tool

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[1] Aka combining [16], pooling, representing etc.

for a completion (extension) of the information provided by a real agent. It is used for aligning domains of processed pds, Prop. 5, as well as for their merging Prop. 6. It and its variant Prop. 7 covering a different scenario provide efficient tools for soft agents' cooperation.

C. Notation, Notions and Basic Relations

$\mathcal{C}$alligraphic fonts mark functionals and $\mathsf{san\ serif}$ fonts denote mappings. Mathfrak superscripts $\mathfrak{o}, \mathfrak{i}, \mathfrak{c}$, and $\mathfrak{e}$ mark optimality, an ideal object, complement and extension, respectively. Small letters, as z, label elements of the underlying deductive DM task[2]. The set of z' is $\{z\}$. Sets are fully specified only if need be. Mostly, they are either subsets of finite-dimensional real space or sets of probability densities (pds). $|\{z\}|$ is the cardinality[3] of $\{z\}$. $\propto$ means proportionality, $\equiv$ stresses the definition by equality. The *action* $a \in \{a\}$, $|\{a\}| > 1$, opted in a one-shot DM, splits the behaviour $b \in \{b\}$ on which DM operates as follows

$$b \equiv (g, a, k) \equiv \tag{1}$$
$$(\text{ignorance during } a \text{ choice, action, knowledge for } a \text{ choice}).$$

The considered DM under uncertainty and incomplete knowledge makes the behaviour a multivariate random variable [3], [21]. This implies that all models are probabilistic distributions given here by their pds. The *behaviour model* b^{a}, a joint pd of behaviours $b \in \{b\}$, depends on the *DM rule*[4] $\mathsf{a} : \{k\} \to \{a\}$. The chain rule for pds [23] factorises the pd

$$\begin{aligned} \mathsf{b}^{\mathsf{a}}(b) &\overbrace{=}^{(1)} \mathsf{b}^{\mathsf{a}}(g, a, k) \\ &= \mathsf{b}^{\mathsf{a}}(g|a, k)\mathsf{b}^{\mathsf{a}}(a|k)\mathsf{b}^{\mathsf{a}}(k) \equiv \mathsf{g}(g|a, k)\mathsf{a}(a|k)\mathsf{k}(k). \end{aligned} \tag{2}$$

The pd g models how the *ignorance* $g \in \{g\}$ depends on the action $a \in \{a\}$ under the given *knowledge* $k \in \{k\}$ modelled by the pd k. The dropping of the superscript $^{\mathsf{a}}$ at g and k respects that the DM rule a influences these pds at most indirectly via the generated actions. The knowledge also includes a description of DM goals. FPD describes them by an *ideal behaviour model*, a pd $\mathsf{b}^{\mathfrak{i}}$ on $\{b\}$. It assigns a large value $\mathsf{b}^{\mathfrak{i}}(b)$ to a desired behaviour $b \in \{b\}$ and a small value $\mathsf{b}^{\mathfrak{i}}(b)$ to an undesired $b \in \{b\}$. The ideal pd $\mathsf{b}^{\mathfrak{i}}$ factorises

$$\begin{aligned} \mathsf{b}^{\mathfrak{i}}(b) &\overbrace{=}^{(1)} \mathsf{b}^{\mathfrak{i}}(g, a, k) \\ &= \mathsf{b}^{\mathfrak{i}}(g|a, k)\mathsf{b}^{\mathfrak{i}}(a|k)\mathsf{b}^{\mathfrak{i}}(k) \equiv \mathsf{g}^{\mathfrak{i}}(g|a, k)\mathsf{a}^{\mathfrak{i}}(a|k)\mathsf{k}^{\mathfrak{i}}(k). \end{aligned} \tag{3}$$

Single action has no impact on the handled knowledge. This motivates the use of the *leave to fate option* (LFO, [14]) for the knowledge model. LFO on $\mathsf{k}^{\mathfrak{i}}$ sets the ideal knowledge model $\mathsf{k}^{\mathfrak{i}}$ equal to the model k, $\mathsf{k}^{\mathfrak{i}} = \mathsf{k}$ on $\{k\}$.

FPD employs relative entropy $\mathcal{R}(\mathsf{y}_{\triangleright}\mathsf{z})$ (RE, aka cross entropy [24] or Kullback-Leibler divergence [25]). RE measures proximity of pds y and z acting on a set $\{z\}$. The RE functional $\mathcal{R}(\mathsf{y}_{\triangleright}\mathsf{z})$ reads

$$\mathcal{R}(\mathsf{y}_{\triangleright}\mathsf{z}) \equiv \int_{\{z\}} \mathsf{y}(z) \ln\left(\frac{\mathsf{y}(z)}{\mathsf{z}(z)}\right) \, \mathrm{d}z = \mathcal{E}\left[\ln\left(\frac{\mathsf{y}}{\mathsf{z}}\right)\right].$$

It is the expectation $\mathcal{E}$, given by the pd y, of $\ln(\mathsf{y}/\mathsf{z})$. Its unusual notation stresses its asymmetry and allows us to indicate the condition in the *conditional RE*

$$\mathcal{R}(\mathsf{y}_{\triangleright}\mathsf{z}|k) \equiv \int_{\{z\}} \mathsf{y}(z|k) \ln\left(\frac{\mathsf{y}(z|k)}{\mathsf{z}(z|k)}\right) \mathrm{d}z = \mathcal{E}\Big[\ln\left(\frac{\mathsf{y}(\cdot|k)}{\mathsf{z}(\cdot|k)}\right)\Big|k\Big].$$

Next Summary 1 fixes notions, defines FPD, extends the notation to meta-tasks and fixes a simplifying assumption on the behaviour-set cardinality.

Summary 1 (FPD; FPD Elements; Finiteness Assumption):

- Tab. I collects the DM elements serving to one-shot FPD.

TABLE I
DM ELEMENTS OF ONE-SHOT FPD

	Meaning	Characterisation
$\{g\}$	an ignorance set	$g \in \{g\}$ is unavailable for an a choice.
$\{a\}$	an action set	The DM options at disposal.
k	a knowledge realisation	The pds g, $\mathsf{g}^{\mathfrak{i}}$, $\mathsf{a}^{\mathfrak{i}}$ are in it. One-shot FPD deals with the realised k, Prop. 1.
g	an ignorance model	The factor of b^{a} that models $g \in \{g\}$.
a	DM rule $\mathsf{a} : k \to \{a\}$	The optimised factor of b^{a}.
$\{\mathsf{a}\}$	a DM-rules set	The optimal DM rule $\mathsf{a}^{\mathfrak{o}}$ is found there.
$\mathsf{g}^{\mathfrak{i}}$	an ideal ignorance model	The factor of $\mathsf{b}^{\mathfrak{i}}$ co-specifies DM goals.
$\mathsf{a}^{\mathfrak{i}}$	an ideal DM rule	The factor of $\mathsf{b}^{\mathfrak{i}}$ co-specifies DM goals.

$$\boxed{\text{The FPD-}\textit{optimal DM rule} \text{ is } \mathsf{a}^{\mathfrak{o}} \in \operatorname{Argmin}_{\mathsf{a} \in \{\mathsf{a}\}} \mathcal{R}(\mathsf{b}^{\mathsf{a}}{}_{\triangleright}\mathsf{b}^{\mathfrak{i}})} \tag{4}$$

- The inductive DM meta-tasks[5] operate on the same types of DM elements as the deductive DM. Capital versions of letters in Tab. I denote them.
- A *slice pd* $\mathsf{s} \in \{\mathsf{s}\}$, a marginal or conditional pd, models behaviour slices $s \in \{s\}$
$$\{b\} = \{s^{\mathfrak{c}}\} \times \{s\}, \ \ \{s^{\mathfrak{c}}\} \text{ complements } \{s\} \text{ to } \{b\}. \tag{5}$$
- Leave to fate option (*LFO*) for a slice s means that for
$$\begin{aligned} \mathsf{b}(b) &= \mathsf{b}(s^{\mathfrak{c}}, s) = \mathsf{b}(s^{\mathfrak{c}}|s)\mathsf{s}(s) \\ \mathsf{b}^{\mathfrak{i}}(b) &= \mathsf{b}^{\mathfrak{i}}(s^{\mathfrak{c}}, s) = \mathsf{b}^{\mathfrak{i}}(s^{\mathfrak{c}}|s)\mathsf{s}^{\mathfrak{i}}(s) \end{aligned} \tag{6}$$
the slice ideal pd $\mathsf{s}^{\mathfrak{i}} = \mathsf{s}$ on $\{s\}$ is opted.

The next assumption and agreement simplify our text.

- DM meta-tasks concern deductive DM tasks with behaviours having a finite amount of realisations[6], $\{b\} = \{g\} \times \{a\} \times \{k\}$ has *cardinality* $|\{b\}| < \infty$.
- The obligatory k-parts g, $\mathsf{g}^{\mathfrak{i}}$, $\mathsf{a}^{\mathfrak{i}}$ are mostly implicit.

Proposition 1 (Solution of One-shot FPD): The FPD-optimal DM rule (4) uses the knowledge realisation k (not the set $\{k\}$). It reads

$$\begin{aligned} \mathsf{a}^{\mathfrak{o}}(a|k) &\propto \mathsf{a}^{\mathfrak{i}}(a|k) \exp[-\mathcal{R}(\mathsf{g}_{\triangleright}\mathsf{g}^{\mathfrak{i}}|a, k)] \\ \mathcal{R}(\mathsf{g}_{\triangleright}\mathsf{g}^{\mathfrak{i}}|a, k) &= \int_{\{g\}} \mathsf{g}(g|a, k) \ln\left(\frac{\mathsf{g}(g|a, k)}{\mathsf{g}^{\mathfrak{i}}(g|a, k)}\right) \, \mathrm{d}g. \end{aligned} \tag{7}$$

[2]DM solved in a deductive way, which exploits complete, inductively created, DM elements.

[3]Mnemonic ties, like $\mathsf{z}(z)$, $z \in \{z\}$, $|\{z\}| < \infty$ expressing that the mapping z operates on the variable z from the set $\{z\}$ of a finite cardinality $|\{z\}|$, are preferred.

[4]Aka an act [21] or a decision function [22]. Specific DM areas use specific names as estimator or control law. This paper proceeds alike.

[5]They mean meta-tasks supporting DM based on inductive reasoning.

[6]The employed smooth numerical representation of preferences' order requires existence of the behaviour-space topology with a countable basis [26]. Thus, the finiteness assumption is unrestrictive and admits unbounded growth to countability. It helps us to avoid subtleties of measure theory and also those of ε-optimality as it guarantees the existence of extremes.

Under LFO (6) for the DM rules, $\mathsf{a}^{\mathsf{i}} = \mathsf{a}$ on $\{a\}$, the optimal DM rule is deterministic. It selects the optimal action $a^{\mathsf{o}} = a^{\mathsf{o}}(k) \in \mathrm{Argmin}_{a\in\{a\}} \mathcal{R}(\mathsf{g}_{\triangleright}\mathsf{g}^{\mathsf{i}}|a,k)$. This action maximises the DM rule a^{o} (7) for the uniform ideal DM rule a^{i}.

Proof It uses chain rule for pds [23], Fubini's theorem on multiple integration [27] and properties RE [25]. For details, see [28], [2] ■

II. Approximation

This section formulates and solves the approximation of a *given* slice pd $\mathsf{s} \in \{\mathsf{s}\}$ by a pd $\hat{\mathsf{s}} \in \{\hat{\mathsf{s}}\}$ as a one-shot FPD meta-task. It specifies the DM elements[7] of this DM meta-task, see Summary 1, and uses Prop. 1. It illustrates a use of the gained DM rule, called *approximator*, and discusses it.

The approximator is designed before observing the behaviour slice $s \in \{s\}$. Thus, the ignorance G is $s \in \{s\} = \{G\}$. The pd $\hat{\mathsf{s}} \in \{\hat{\mathsf{s}}\}$, approximately modelling slices $s \in \{s\}$, is opted. Thus, the action A is $\hat{\mathsf{s}} \in \{\hat{\mathsf{s}}\} = \{A\}$. The knowledge K contains the approximated pd s and the set $\{\hat{\mathsf{s}}\}$ of approximations. The approximators are conditional pds

$$\mathsf{A} \in \{\mathsf{A}\} \equiv \{\mathsf{A}(\hat{\mathsf{s}}|K) : (K \to A) \equiv ((\mathsf{s}, \{\hat{\mathsf{s}}\}) \to \hat{\mathsf{s}})\}.$$

The ignorance model within this FPD meta-task has the unambiguous form

$$\mathsf{G}(G|A,K) = \mathsf{G}(s|\hat{\mathsf{s}}, (\mathsf{s}, \{\hat{\mathsf{s}}\})) = \mathsf{s}(s), \quad G = s \in \{s\}. \tag{8}$$

It respects that the approximator $\hat{\mathsf{s}} \in \{\hat{\mathsf{s}}\}$ and the implicitly-present ideal factors $\mathsf{G}^{\mathsf{i}}, \mathsf{A}^{\mathsf{i}}$ have no influence on slice realisations $s \in \{s\}$, while the given pd s models them by its definition. The ideal ignorance model, a factor of the meta-version of (3), is also unambiguous

$$\mathsf{G}^{\mathsf{i}}(G|A,K) = \mathsf{G}^{\mathsf{i}}(s|\hat{\mathsf{s}},K) = \hat{\mathsf{s}}(s), \quad G = s \in \{s\}. \tag{9}$$

It expresses the primary goal of this meta-task: to choose the pd $\hat{\mathsf{s}} \in \{\hat{\mathsf{s}}\}$, which ideally describes slices $s \in \{s\}$. The ideal approximator $\mathsf{A}^{\mathsf{i}}(\hat{\mathsf{s}}|K)$, a factor in the meta-version of (3), has to guarantee the existence of the FPD-optimal approximator. Thus, its *support* $\mathrm{supp}\left[\mathsf{A}^{\mathsf{i}}\right] \equiv \{\hat{\mathsf{s}} : \mathsf{A}^{\mathsf{i}}(\hat{\mathsf{s}}|K) > 0\}$ has to fulfill

$$\mathrm{supp}\left[\mathsf{A}\right] = \{\hat{\mathsf{s}}\} \subset \mathrm{supp}\left[\mathsf{A}^{\mathsf{i}}\right]. \tag{10}$$

Proposition 2 (Approximation Principle): The FPD-optimal approximator, using the knowledge $K = (\mathsf{s}, \{\hat{\mathsf{s}}\})$, is the pd

$$\mathsf{A}^{\mathsf{o}}(\hat{\mathsf{s}}|K) \quad \propto \quad \mathsf{A}^{\mathsf{i}}(\hat{\mathsf{s}}|K)\exp[-\mathcal{R}(\mathsf{s}_{\triangleright}\hat{\mathsf{s}})], \quad \hat{\mathsf{s}} \in \{\hat{\mathsf{s}}\}.$$

Under LFO (6) for the approximators $\mathsf{A}^{\mathsf{i}} = \mathsf{A}$ on $\{\hat{\mathsf{s}}\}$, the FPD-optimal approximator is deterministic and chooses

$$\hat{\mathsf{s}}^{\mathsf{o}} \in \mathrm{Arg}\min_{\hat{\mathsf{s}}\in\{\hat{\mathsf{s}}\}} \mathcal{R}(\mathsf{s}_{\triangleright}\hat{\mathsf{s}}) = \mathrm{Arg}\min_{\hat{\mathsf{s}}\in\{\hat{\mathsf{s}}\}} \int_{\{s\}} \mathsf{s}(s)\ln\left(\frac{\mathsf{s}(s)}{\hat{\mathsf{s}}(s)}\right)\,\mathrm{d}s. \tag{11}$$

Proof Prop. 1 for (8)-(10) gives the solution as $\mathcal{R}(\mathsf{s}_{\triangleright}\hat{\mathsf{s}}) = \mathcal{R}(\mathsf{s}_{\triangleright}\hat{\mathsf{s}}|\hat{\mathsf{s}}, (\mathsf{s}, \{\hat{\mathsf{s}}\}))$. ■

[7]The omission of the "meta-" prefix is harmless due to the capitals use.

A. Use of Approximation Principle

This part applies the approximation principle, Prop. 2, to recursive approximate learning. The presentation first outlines how a non-standard universally approximating observation model arises. Then, Bayes' rule "naturally" appears as the proper learning tool and the need for its approximation arises. Then, the used set of approximating models and the approximation principle dictate the approximation way.

Learning extends the knowledge k by the *data* $d = (o, a) \in \{d\} = \{o\} \times \{a\}$. The *observation* $o \in \{o\}$ is made on the closed DM loop after applying the action $a \in \{a\}$. Learning needs an observation model, the pd o relating o to (a, k). The next sketch motivates its widely useable parametric form.

The set $\{d\}$ can be covered by a countable amount of open subsets, see Note 6. The logarithm of any smooth data pd can be arbitrarily-well approximated by linear expansions on a finite selection of these subsets (indexed by $j \in \{j\}$). This indicates the universal approximation property [29] of the non-standard finite-mixture ratio [30], $|\{j\}| < \infty$,

$$\mathsf{o}(o|a,p) = \frac{\sum_{j\in\{j\}} \exp\left\langle \mathsf{u}_j(o,a), \mathsf{w}_j(p_j)\right\rangle_j}{\sum_{j\in\{j\}} \int_{\{o\}} \exp\left\langle \mathsf{u}_j(o,a), \mathsf{w}_j(p_j)\right\rangle_j \,\mathrm{d}o}. \tag{12}$$

There, $\mathsf{w}_j(p_j)$ are basis functions used for the expansion within the jth subset. They are parameterised by a finite dimensional parameter[8] $p_j \in \{p_j\}$. The expansion coefficients are $\mathsf{u}_j(o,a)$ and the real-valued mappings $\langle \mathsf{u}, \mathsf{w}\rangle_j$ are linear in u. This outline indicates richness of the parametric model

$$\mathsf{o}(o|a,p) = \mathsf{f}(\exp\left\langle \mathsf{u}(o,a), \mathsf{w}(p)\right\rangle). \tag{13}$$

It generalises (12) to a smooth function f and denotes $\mathsf{w}(p) = (\mathsf{w}_j(p_j))_{j\in\{j\}}$, $p = (p_j)_{j\in\{j\}}$, $\mathsf{u} = (\mathsf{u}_j(o,a))_{j\in\{j\}}$, and $\langle\cdot,\cdot\rangle = (\langle\cdot,\cdot\rangle)_{j\in\{j\}}$. The adequate, case-dependent, value of the parameter $p \in \{p\}$ is a priori unknown. Its presence extends the ignorance to the pair $g =$ (unused observation o, unknown parameter $p \in \{g\} = \{o\} \times \{p\}$. The ignorance model factorises[9]

$$\begin{aligned} \mathsf{g}(g|a) = \mathsf{g}(o,p|a) &= \mathsf{o}(o|a,p) \times \mathsf{p}(p) \qquad (14)\\ &= \text{parametric observation model} \times \textit{prior pd} \text{ of parameter.} \end{aligned}$$

The knowledge k determines the parametric observation model of a fixed functional form (13) and is to provide the prior pd $\mathsf{p}(p)$ (14). The learning updates the prior pd $\mathsf{p}(p)$ to the *posterior pd* $\mathsf{p}(p|d)$. Conditioning by the observed data d, called Bayes' rule, deductively provides this updating

$$\mathsf{p}(p|d) \propto \mathsf{g}(g|a) = \mathsf{o}(o|a,p) \times \mathsf{p}(p). \tag{15}$$

It is well-implementable when the function f (13) is identity

$$\mathsf{o}(o|a,p) = \exp\left\langle \mathsf{u}(o,a), \mathsf{w}(p)\right\rangle, \tag{16}$$

[8]The discussion may apply to a hidden evolving state. It leads to stochastic filtering area. The exposition simplicity makes us to deal with the parameter estimation only. It also relies an implicit presence of the knowledge k.

[9]The equality $\mathsf{p}(p|a) = \mathsf{p}(p)$ reflects the assumption (natural conditions of control [23]) that the parameter p belongs to ignorance g and is unused by the DM rule yielding $a \in \{a\}$.

when these models belong to the exponential family (*EF*, [31]). They admit the feasible conjugate prior pds

$$\mathsf{p}(p) = \mathsf{p}(p|v) \equiv \frac{\exp\langle v, \mathsf{w}(p)\rangle}{\mathcal{J}(v)}, \quad \text{where } v \in \boldsymbol{\{}v\boldsymbol{\}} \tag{17}$$
$$\boldsymbol{\{}v\boldsymbol{\}} \quad = \quad \{v\text{'s in the range of } \mathsf{u} \text{ giving } \ \mathcal{J}(v) \in (0,\infty)\}.$$

Bayes' rule preserves the form (17) of the conjugate pd. This reduces the functional updating (15) to the algebraic one

$$v(d) = v + \mathsf{u}(d), \quad d = (o, a). \tag{18}$$

EF practically exhausts observation models reducing Bayes' rule to algebraic updating (18) of values of a finite-dimensional sufficient statistic [32]. EF is, however, quite narrow. In the generic case (13), the prior pd $\mathsf{p}(p)$ of the form (17) maps by (15) on the $\mathsf{p}(p|d)$ of another form. The wish to stick with a feasible algebraic updating calls for an approximation.

The unknown parameter p plays the role of behaviour slice $p = s$. Its posterior pd $\mathsf{p}(p|d) = \mathsf{s}(s)$ (15) is known. Pds of the form (17) offer as its approximation $\hat{\mathsf{p}}(p|v(d)) = \hat{\mathsf{s}}(s)$. Prop. 2 provides the optimal approximator.

Proposition 3 (Moment Fitting): Under LFO for approximators, the optimal ("point") approximation (11) of the posterior pd $\mathsf{p}(p|d)$ among pds $\hat{\mathsf{p}}(p|v(d))$ of the conjugate form (17) is given by the optimal statistic value

$$v^{\mathsf{o}}(d) \in \operatorname{Arg}\min_{v\in\boldsymbol{\{}v\boldsymbol{\}}} \left[\ln(\mathcal{J}(v)) - \left\langle v, \int_{\boldsymbol{\{}p\boldsymbol{\}}} \mathsf{p}(p|d)\mathsf{w}(p)\ \mathrm{d}p \right\rangle\right].$$

Proof A simple direct use of Prop. 2 is omitted. ■

The found $v^{\mathsf{o}}(d)$ optimally weights the moments, the expectation of $\mathsf{w}(p)$ with respect to $\mathsf{p}(p|d)$.

B. Discussion of the Approximation Principle

The choice of the ideal approximator A^{i} is the only freedom left in the approximation task. The pd A^{i}, restricted now by (10), can express, for instance, preferences for simpler approximations. This newly recognised possibility is yet unelaborated and the approximation (11), gained under LFO, dominates. It was derived in [33] via a different reasoning while employing more restrictive assumptions. All other meta-DM elements have no meaningful alternatives.

Importantly, the approximation principle recommends $\mathcal{R}(\mathsf{s}_{\triangleright}\hat{\mathsf{s}})$ as the preferred proximity measure. The subsequent DM meta-tasks heavily depend on this choice. Both the use of $\mathcal{R}$ and the order of $\mathcal{R}$-arguments matter. The arguments' order is opposite to that in the popular variational Bayes' approximation [34]. The evaluation feasibility, balancing the "improper order", is the decisive argument for its extensive use. It may be dearly paid by the reached quality as the following example demonstrates.

a) Influence of the RE-arguments order: Monte Carlo simulation generated 10000 samples of joint pds s of data pairs $s = (o, a)$, each having 3 possible values. In the standard version, values $\mathsf{s}(a, b)$ were sampled from uniform pd on the interval $[0, 1]$. In the difficult version, just the value $\mathsf{s}(1, 1) = 1e-6$ was enforced. In both cases, values s were properly normalised. Each joint pd $\mathsf{s}(o, a)$ was approximated by pds with independent o and a, i.e. $\hat{\mathsf{s}}(o, a) = \hat{\mathsf{o}}(o)\hat{\mathsf{a}}(a)$. The approximation principle, Prop. 2, recommends $\hat{\mathsf{o}}$, $\hat{\mathsf{a}}$ be marginal pds of s. The opposite order of RE leads to standard equations of the variational Bayes. They can be simply solved to get $\hat{\mathsf{s}}(o, a) = \hat{\mathsf{o}}(o)\hat{\mathsf{a}}(a)$. Fig. 1 shows histograms of absolute approximation errors

$$\sum_{o\in\boldsymbol{\{}o\boldsymbol{\}}, a\in\boldsymbol{\{}a\boldsymbol{\}}} \operatorname{abs}(\mathsf{s}(o, a) - \hat{\mathsf{s}}(o, a)). \tag{19}$$

This loss judges the deviation of the expected utility caused by the approximation. It prefers none of the compared ways.

In the difficult simulation version, the order implied by the approximation principle visibly outperforms the alternative order. In the standard simulation version, the approximation quality is similar for both argument orders.

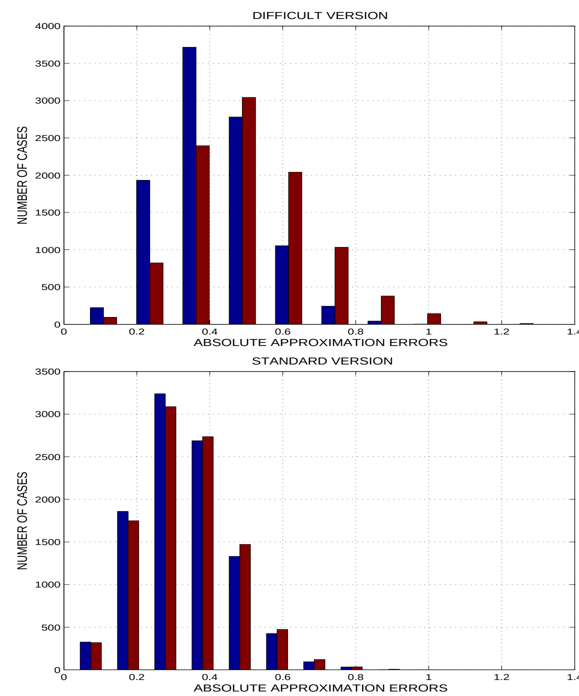

Fig. 1. Histograms of absolute approximation errors (19). Blue columns correspond to the order implied by the approximation principle, red ones to the opposite order. The upper figure corresponds with the difficult version specific by $\mathsf{s}(1, 1) \approx 0$. The lower figure shows standard version with entries of s mostly well within (0,1). ■

Let us stress that the example resulting into Prop. 3 confirms the desirable methodological parsimony: the processed DM elements vary, not the processing method. For instance, finite mixtures of Dirac's type pds serving as the set of approximating pds result in the updating used in Monte-Carlo based estimation [35]. The parsimony removes the danger brought by an ad hoc choice of the inductive method. The use of deductive DM meta-tasks guarantees that possible *unsatisfactory results are not caused by a wrong choice of the inductive method.*

III. Extension

This section formulates and solves the extension of an *incomplete information about the slice pds* $\mathsf{s} \in \boldsymbol{\{}\mathsf{s}\boldsymbol{\}}$ acting on $\boldsymbol{\{}s\boldsymbol{\}}$ as a one-shot FPD meta-task. The incompleteness means that the slice pd belongs to a set $\boldsymbol{\{}\mathsf{s}\boldsymbol{\}}$ containing more than one element. Similarly as in Sec. II, the solution selects appropriate meta-DM elements, see Summary 1, and uses Prop. 1. The presentation illustrates uses of the gained DM rule, called *extensor*. It discusses extensors both generally and in connection with their respective applications.

The extensor is designed before observing the behaviour slice $s \in \{s\}$. Thus, the ignorance G is $s \in \{s\} = \{G\}$. The extension $\mathsf{s} \in \{\mathsf{s}\}$ of the agent's information, a pd on $s \in \{s\}$, has to be opted. Thus, the action A is the pd $\mathsf{s} \in \{\mathsf{s}\} = \{A\}$. The optimised extensors are conditional pds in the set

$$\mathsf{A} \in \{\mathsf{A}\} \equiv \{\mathsf{A}(\mathsf{s}|K) : \operatorname{supp}[\mathsf{A}] = \{\mathsf{s}\}\}. \tag{20}$$

The ignorance model has the unambiguous form

$$\mathsf{G}(G|A, K) = \mathsf{G}(s|\mathsf{s}, K) = \mathsf{s}(s) \tag{21}$$

as the opted $A = \mathsf{s}$ models the slice $s \in \{s\}$ by its definition. The ideal ignorance model is an, a priori best, slice model s^{i}

$$\mathsf{G}^{\mathsf{i}}(G|A, K) = \mathsf{G}^{\mathsf{i}}(s|\mathsf{s}, K) = \mathsf{s}^{\mathsf{i}}(s). \tag{22}$$

It respects that the a priori best choice the slice description s^{i} is uninfluenced by the chosen extension $A = \mathsf{s}$. Generically $\mathsf{s}^{\mathsf{i}} \notin \{\mathsf{s}\}$. The ideal extensor A^{i} has to guarantee existence of the FPD-optimal extensor, see Prop. 1,

$$\{\mathsf{s}\} \subset \operatorname{supp}\left[\mathsf{A}^{\mathsf{i}}\right]. \tag{23}$$

Proposition 4 (Extension by MRE Principle,): The FPD-optimal extensor A^{o}: (a) describing the extensions $\mathsf{s} \in \{\mathsf{s}\}$, which contains slice pds compatible with the agent's information about them, and (b) reflecting the a priori chosen s^{i} and A^{i} with $\{\mathsf{s}\} \subset \operatorname{supp}\left[\mathsf{A}^{\mathsf{i}}\right]$, reads

$$\mathsf{A}^{\mathsf{o}}(\mathsf{s}|K) \quad \propto \quad \mathsf{A}^{\mathsf{i}}(\mathsf{s}|K) \exp[-\mathcal{R}(\mathsf{s}_{\triangleright}\mathsf{s}^{\mathsf{i}})]. \tag{24}$$

Under LFO (6) for extensors, $\mathsf{A}^{\mathsf{i}} = \mathsf{A}$, the FPD-optimal extensor is deterministic. It chooses the optimal extension

$$\mathsf{s}^{\mathsf{o}} \in \operatorname{Arg}\min_{\mathsf{s} \in \{\mathsf{s}\}} \mathcal{R}(\mathsf{s}_{\triangleright}\mathsf{s}^{\mathsf{i}}|\mathsf{s}, \mathsf{s}^{\mathsf{i}}) \equiv \operatorname{Arg}\min_{\mathsf{s} \in \{\mathsf{s}\}} \mathcal{R}(\mathsf{s}_{\triangleright}\mathsf{s}^{\mathsf{i}}). \tag{25}$$

The choice (25) is known as the *minimum relative-entropy (MRE) principle* [24]. Under LFO (6) for ignorance models, $\mathsf{s}^{\mathsf{i}} = \mathsf{s}$, the FPD-optimal extensor is

$$\begin{aligned} \mathsf{A}^{\mathsf{o}} \quad &\in \quad \operatorname{Arg}\min_{\mathsf{A} \in \{\mathsf{A}\}} \mathcal{R}(\mathsf{A}_{\triangleright}\mathsf{A}^{\mathsf{i}}) \\ \{\mathsf{A}\} \quad &= \quad \{\mathsf{A} = \mathsf{A}(\mathsf{s}|K) : \operatorname{supp}[\mathsf{A}] = \{\mathsf{s}\}\}. \end{aligned} \tag{26}$$

The choice (26) is called the *generalised MRE principle* [20]. **Proof** Prop. 1 applied with (21) – (23), directly solves the extension task. ■

A. Discussion of MRE Principle

The examples that follow indicate the potential of the MRE principle. Each has its discussion so that this subsection makes just general comments.

The MRE principle (25) was derived in [24] by a different technique. Its specific case with uniform s^{i} is known as the maximum entropy principle [36].

The generalised MRE principle (26) was derived in a slightly different way in [20]. The neglected version (24) is the novel outcome of the current "polishing".

The set $\{\mathsf{s}\}$ is the vital part of the processed knowledge K. The smaller $\{\mathsf{s}\}$ is, while still compatible with the knowledge supplied by the agent, the better. The ideal-defining pds s^{i}, A^{i} are only free options of the MRE principle. Specific, but still general cases, Sec. III-B, reduce even this freedom.

The opted extension s is the first argument of RE unlike the opted approximation $\hat{\mathsf{s}}$ of the known pd, Prop. 2. It equips a RE use with the rule of thumb

$$\boxed{\text{The 1st argument of RE is believed to describe reality.}} \tag{27}$$

B. Use of MRE Principle

This part illustrates the MRE-principle potential and gives practical results.

1) Modelling: Modelling dominates the use of the MRE principle. A domain knowledge[10] provides, mostly deterministic, approximate relations of the ignorance to action and knowledge realisations. The need to extend this agent's information to the probabilistic environment model was the primary impetus leading to the maximum entropy principle [36] and the MRE principle [24]. A typical problem and its solution follow. It reveals general, rarely addressed, problems stemming from non-identical domains of treated pds and their incompatibility.

The modelling, cf. Sec. II-A, deals with the ignorance $g = (o, p) =$ (unused observation, unknown parameter) $\in \{g\} = \{o\} \times \{p\}$. The ignorance model (14) factorises $\mathsf{g}(g|a, k) = \mathsf{g}(o, p|a, k) = \mathsf{o}(o|p, a, k) \times \mathsf{p}(p|k) =$ observation model $\times$ prior pd. The observation $o \in \{o\}$ is scalar. The vector case boils down to scalar-observation models via the chain rule.

The processed information relating the observation to the opted action and the employed knowledge has the typical form

$$o - \mathsf{f}_j(p_j, a_j, k_j) \approx 0, \quad j \in \{j\}, \ |\{j\}| < \infty. \tag{28}$$

The agent provides the functions $(\mathsf{f}_j)_{j \in \{j\}}$. They depend on finite-dimensional, instance-dependent, unknown parameters $(p_j \in \{p\}_j)_{j \in \{j\}}$. The parameter p_j may also characterise deviations, which make the equality (28) approximate.

For any fixed $j \in \{j\}$ and (p_j, a_j, k_j), the observation o is the modelled slice. The set of observation models o compatible with (28) is the set $\{\mathsf{o}\}$ of slice pds. The relation (28), taken as equality with deviations of zero expectation, gives

$$\begin{aligned} \{\mathsf{o}\} \quad \equiv \quad &\Big\{\mathsf{o}(o|p_j, a_j, k_j) : \int_{\{o\}} o\mathsf{o}(o|p_j, a_j, k_j) \mathrm{d}o \\ &= \mathcal{E}[o|p_j, a_j, k_j] = \mathsf{f}_j(p_j, a_j, k_j)\Big\}. \end{aligned} \tag{29}$$

With a chosen ideal observation model[11] $\mathsf{o}^{\mathsf{i}}_j$, MRE (25) has the solution [36]

$$\mathsf{o}^{\mathsf{o}}_j(o|p_j, a_j, k_j) \propto \mathsf{o}^{\mathsf{i}}_j(o|p_j, a_j, k_j) \exp[-o\lambda(p_j, a_j, k_j)]. \tag{30}$$

There, the Lagrangian multiplier $\lambda(p_j, a_j, k_j)$ meets the constraint (29). The pd (30) is also the mode of the extensor (24) with the uniform ideal extensor.

[10] It means knowledge originating in theoretical or experimental physics, chemistry, technology, economy, biology, social sciences, etc.

[11] For instance, the uniform pd covering tightly the set $\{o\}$ of observations o and having the expectation equal $\mathsf{f}_j(p_j, a_j, k_j)$.

The solution applies $\forall j \in \{j\}$. Mapping of the pds $(\mathsf{o}^{\mathsf{o}}_j)_{j\in\{j\}}$ (30) on single model respecting agent-supplied information (28) is then the crucial final modelling step. The solution of this *merging problem* is in Sec. III-B5.

2) Discussion of Modelling: The outlined modelling is powerful as it may combine theoretical, data-based, semi-experimental or expert supplied information. Merging of the gained models is hard due to the next obstacles.

a) Problem of Different Domains: A jth knowledge source, typically another agent, operates on data $d_j = (o, a_j) \in \{d_j\} = \{o\} \times \{a_j\}$. As a rule, $\{a_j\} \neq \{a_\iota\}$ for $j \neq \iota$. Merging of observation models $\mathsf{o}^{\mathsf{o}}_j$ and $\mathsf{o}^{\mathsf{o}}_\iota$ is non-trivial and rarely explicitly addressed.

b) Incompatibility Problem: Even when domains of sources $j \neq \iota$ are the same, their supports may differ up to $\mathrm{supp}\,[\mathsf{o}_j] \cap \mathrm{supp}\,[\mathsf{o}_\iota] = \emptyset$. This incompatibility is more the rule rather than the exception and a priori it is rarely avoidable.

Both obstacles are also faced when merging knowledge about unknown parameters, within the knowledge elicitation [7], [8], or when merging pds for cooperation purposes [13], [15]. They drive the next domain extension and merging.

3) Domain Extension: Merging of pds, whose need is revealed in Sec. III-B1, is to cope with different domains of pds to be merged, Sec. III-B2. The section resolves this.

A finite collection of pds indexed by $j \in \{j\}$ is considered. A jth pd to be merged s_j operates on the behaviour slice $s_j \in \{s_j\}$. Generically,

$$\{s_j\} \subsetneq \{s^{\mathfrak{e}}\} \equiv \cup_{j\in\{j\}}\{s_j\}. \quad (31)$$

Each pd s_j, with the domain $\{s_j\}$, has to be extended on a pd $\mathsf{s}^{\mathfrak{e}}_j$ modelling slices $s^{\mathfrak{e}} \in \{s^{\mathfrak{e}}\}$ (31). For the pd $\mathsf{s} = \mathsf{s}_j$ with a fixed $j \in \{j\}$, the analogy of (5) with $s^{\mathfrak{e}} = (s^{\mathfrak{c}}, s)$ gives the set $\{\mathsf{s}^{\mathfrak{e}}\}$ of possible extensions $\mathsf{s}^{\mathfrak{e}}$ of the given pd s on $\{s\}$

$$\{\mathsf{s}^{\mathfrak{e}}\} \equiv \left\{\mathsf{s}^{\mathfrak{e}}(s^{\mathfrak{e}}) = \mathsf{s}^{\mathfrak{e}}(s^{\mathfrak{c}}, s) = \mathsf{s}^{\mathfrak{e}}(s^{\mathfrak{c}}|s)\mathsf{s}(s)\right\}. \quad (32)$$

This set is the only but vital specific choice in the construction of the extensor called *domain extensor*. The next application of Prop. 4 provides it.

Proposition 5 (Optimal Domain Extensor): With the given ideal domain extensor A^{i}, meeting (23), and the pd s^{i}, determining the ideal ignorance model (22), the MRE principle provides the optimal domain extensor of the form (24).

Under LFO (6) for the domain extensor, $\mathsf{A}^{\mathsf{i}} = \mathsf{A}$ on $\{\mathsf{s}\}$, the optimal domain extension (25) is deterministic and reads

$$\mathsf{s}^{\mathsf{o}}(s^{\mathfrak{c}}, s) = \frac{\mathsf{s}^{\mathsf{i}}(s^{\mathfrak{c}}, s)}{\int_{\{s\}^{\mathfrak{c}}} \mathsf{s}^{\mathsf{i}}(s^{\mathfrak{c}}, s)\, \mathrm{d}s^{\mathfrak{c}}}\mathsf{s}(s) = \mathsf{s}^{\mathsf{i}}(s^{\mathfrak{c}}, s)\frac{\mathsf{s}(s)}{\mathsf{s}^{\mathsf{i}}(s)}. \quad (33)$$

This domain extension maximises the domain extensor (24) for the uniform A^{i}.

Proof It directly follows from assumptions on involved DM elements and the fact that RE reaches its minimum for equal arguments. ■

4) Discussion of Domain Extension: The optimal domain extension $\mathsf{s}^{\mathsf{o}}(s^{\mathfrak{c}}, s)$ (33) replaces in the pd $\mathsf{s}^{\mathsf{i}}(s^{\mathfrak{c}}, s)$ its marginal pd $\mathsf{s}^{\mathsf{i}}(s)$ by the given pd $\mathsf{s}(s)$. The result was derived in [13].

Mostly, the ideal pd $\mathsf{s}^{\mathsf{i}} \notin \{\mathsf{s}^{\mathfrak{e}}\}$ (32) as its marginal $\mathsf{s}^{\mathsf{i}}(s) \neq \mathsf{s}(s)$ on $s \in \{s\}$. This freedom is vital as the merging, Sec. III-B5, deals jointly with the sets (32) assigned to the respective pds $(\mathsf{s}_j)_{j\in\{j\}}$ and we have to cope with the fact that mostly $\mathsf{s}_j \neq \mathsf{s}_\iota$ for $j \neq \iota$.

The ideal domain-extensor A^{i} is to have support covering $\{\mathsf{s}^{\mathfrak{e}}\}$, cf. (20). No other general guide of its choice has been recognised. LFO $\mathsf{A}^{\mathsf{i}} = \mathsf{A}$ used in Prop. 5 seems to be better than the insufficient-reasons-motivated uniform choice.

The ideal domain extension s^{i} has a universal choice in merging. Its description in Sec. III-B7 exploits the solution of the merging problem, see Sec. III-B5.

5) Merging of PDs: Modelling, Sec. III-B1, calls for merging of a finite collection of given pds

$$\mathsf{s}^{|\{j\}|} \equiv (\mathsf{s}_j)_{j\in\{j\}} \text{ having a common domain } \{s\}. \quad (34)$$

Prop. 5 makes this assumption unrestrictive.

The pd s_j originates in jth information source, cf. Sec. III-B1, interpreted as the jth agent. The addressed *merging aims to extract the information provided by all agents to benefits of an agent* $\iota \in \{j\}$. The built DM rule, *merger* $\mathsf{A}(\mathsf{s}|K)$, offers to the ιth agent the merged pd s on $\{s\}$. The merger knowledge K includes pds (34). The merged pd s is the action $A \in \{A\}$ opted before observing a slice $s \in \{s\}$, which forms the ignorance G of this DM meta-task.

The formulation fits the MRE principle, Prop. 4. It needs to choose the set $\{\mathsf{s}\} = \{A\}$ of the merged pds s, the ideal merger A^{i} and the ideal pd s^{i}.

a) The choice of $\{\mathsf{s}\} = \{A\}$*:* A merged pd s extracts well the information in pds $\mathsf{s}^{|\{j\}|}$ (34) iff it well approximates all of them. The approximation principle, Prop. 2, implies the ιth agent expects that the optimal merger prefers small values $(\mathcal{R}(\mathsf{s}_{j\triangleright}\mathsf{s}))_{j\in\{j\}}$. This motivates the choice of the set of well-merged pds

$$\{\mathsf{s}\} \equiv \{\mathsf{s} : \mathcal{R}(\mathsf{s}_{j\triangleright}\mathsf{s}) \leq \gamma_j,\ \forall j \in \{j\}\} \quad (35)$$

with $(\gamma_j)_{j\in\{j\}}$ as small as possible. This verbally expressed vectorial wish converts into the scalar optimisation parameterised by a probabilistic vector $\mathsf{c} = (\mathsf{c}_j)_{j\in\{j\}} \in \{\mathsf{c}\}$ and by an optional $\kappa > 0$

$$\gamma_j \equiv \mathsf{c}_j\kappa. \quad (36)$$

The known pd c expresses the creed of ιth agent into information provided by respective agents. It can be a priori chosen as uniform pd and gradually learnt in the Bayesian way[12].

The set (35) shrinks with the optional κ. The smallest κ^{o}, which keeps $\{\mathsf{s}\} \neq \emptyset$ is clearly preferred. The desired mergers A have supports covering $\{\mathsf{s}\}$. Thus, their set $\{\mathsf{A}\}$ meets

$$\{\mathsf{A}\} \subset \left\{\mathsf{A} : \int_{\{s\}} \mathsf{A}(\mathsf{s})\mathcal{R}(\mathsf{s}_{j\triangleright}\mathsf{s})\, \mathrm{d}\mathsf{s} \leq \mathsf{c}_j\kappa, \forall j \in \{j\}\right\}. \quad (37)$$

b) The choice of the ideal mergers A^{i}*:* It is chosen as Dirichlet's pd [37] $\mathsf{A}^{\mathsf{i}} = \mathsf{D}(\mathsf{s}|\mathsf{v}^{\mathsf{i}})$. This simplifies evaluations

[12] All treated DM elements should have a pointer to the agent to which the merging serves. Its simplifying omission has to be kept in mind. The merging of the same information by another agent gives a different merger as it uses a different (private) creed pd c.

without a generality loss[13] [40]. The ideal degrees of freedom $\mathsf{v}^{\mathsf{i}} = (\mathsf{v}^{\mathsf{i}}(s) > 0)_{s\in\{s\}}$ parameterise the pd $\mathsf{D}(\mathsf{s}|\mathsf{v}^{\mathsf{i}})$ as follows

$$\mathsf{D}(\mathsf{s}|\mathsf{v}^{\mathsf{i}}) \propto \prod_{s\in\{s\}} [\mathsf{s}(s)]^{\mathsf{v}^{\mathsf{i}}(s)-1}. \quad (38)$$

The ideal degrees of freedom v^{i} of the pd (38) can be expressed as a scaled slice pd s_0, $\mathsf{v}^{\mathsf{i}}(s) = \lambda_0 \mathsf{s}_0(s)$, $\lambda_0 > 0$. Then the pd s_0 enters evaluations exactly as the pds in $\mathsf{s}^{|\{j\}|}$. By including s_0 into the collection $\mathsf{s}^{|\{j\}|}$ (34), the ideal merger coincides with the improper Dirichlet' pd $\mathsf{D}(\mathsf{s}|\mathsf{v}^{\mathsf{i}} \to 0^+)$.

Prop. 6, describing below the optimal merger A^{o}, uses digamma function ψ, derived from Euler's gamma function $\Gamma(z) \equiv \int_0^\infty x^{z-1}\exp(-x)\,\mathrm{d}x$, [41],

$$\psi(z) \equiv \frac{\mathrm{d}\ln(\Gamma(z))}{\mathrm{d}z}, \quad z > 0. \quad (39)$$

The proposition proof wields the formula [14] for $s \in \{s\}$

$$\int_{\{\mathsf{s}\}} \mathsf{D}(\mathsf{s}|\mathsf{v}) \ln(\mathsf{s}(s))\,\mathrm{d}\mathsf{s} \;=\; \psi(\mathsf{v}(s)) - \psi\Big(\sum_{s\in\{s\}} \mathsf{v}(s)\Big) \quad (40)$$

and the *entropy* definition[14] $\mathcal{H}(\mathsf{s}) \equiv -\sum_{s\in\{s\}} \mathsf{s}(s)\ln(\mathsf{s}(s))$.

c) The choice of s^{i}*:* LFO $\mathsf{s}^{\mathsf{i}} = \mathsf{s}$ is used. Nothing is lost as it suffices to include s^{i} into the collection of pds $\mathsf{s}^{|\{j\}|}$ (34) if the agent has a firm idea about s^{i}.

Proposition 6 (Merging via MRE Principle): Let: (a) the creed pd c in (36) be given; (b) the improper conjugate ideal merger $\mathsf{A}^{\mathsf{i}}(\mathsf{s}|K) = \mathsf{D}(\mathsf{s}|\mathsf{v}^{\mathsf{i}} \to 0^+)$ be used; (c) LFO (6) be applied to the ignorance model, $\mathsf{s}^{\mathsf{i}} = \mathsf{s}$; (d) the set $\{\mathsf{s}\} \neq \emptyset$ (35) with the active bounds $\gamma_j = \mathsf{c}_j\kappa$ (36) and the smallest κ be considered; (e) the knowledge-expressing pds $\mathsf{s}^{|\{j\}|}$, defined on common slices $s \in \{s\}$ but having arbitrary supports, be merged. Then, the optimal merger $\mathsf{A}^{\mathsf{o}}(\mathsf{s}) = \mathsf{D}\Big(\mathsf{s}\Big|\mathsf{v}^{\lambda^{\mathsf{o}}}\Big)$ respects (37). It is given by, see (39), (40),

$$\begin{aligned}
\mathsf{v}^{\lambda^{\mathsf{o}}} &\equiv \Big(\mathsf{v}^{\lambda^{\mathsf{o}}}(s)\Big)_{s\in\{s\}}, \quad \mathsf{v}^{\lambda}(s) \equiv \sum_{j\in\{j\}} \lambda_j \mathsf{s}_j(s) \quad \text{with} \\
\lambda^{\mathsf{o}} &\in \operatorname{Arg}\min_{\lambda\in\{\lambda\}} \Big[\psi\Big(\sum_{j\in\{j\}} \lambda_j\Big) - \sum_{s\in\{s\}} \mathsf{s}_1(s)\psi(\mathsf{v}^{\lambda}(s))\Big].
\end{aligned}$$

The set $\{\lambda\}$ contains $|\{j\}|$-vectors with entries $(\lambda_j \geq 0)_{j\in\{j\}}$ meeting, $\forall j \in \{j\}$, $\mathsf{c}_1\mathcal{H}(\mathsf{s}_j) - \mathsf{c}_j\mathcal{H}(\mathsf{s}_1) =$

$$(\mathsf{c}_1 - \mathsf{c}_j)\psi\Big(\sum_{j\in\{j\}} \lambda_j\Big) + \sum_{s\in\{s\}} [\mathsf{c}_j\mathsf{s}_1(s) - \mathsf{c}_1\mathsf{s}_j(s)]\psi\left(\mathsf{v}^{\lambda}(s)\right).$$

Proof The wish to make bounds in (35) as tight as possible activates at least some constraints. The same holds for (37). Minimisation respecting (37) reduces to that of the Kuhn-Tucker functional given by multipliers $\lambda = (\lambda_j \geq 0)_{j\in\{j\}}$

$$\begin{aligned}
&\operatorname{Arg}\min_{\mathsf{A}\in\{\mathsf{A}\}} \Big[\mathcal{R}(\mathsf{A}_{\triangleright}\mathsf{A}^{\mathsf{i}}) + \sum_{j\in\{j\}} \lambda_j \int_{\mathsf{s}} \mathsf{A}(\mathsf{s})\mathcal{R}(\mathsf{s}_{j\triangleright}\mathsf{s})\mathrm{d}\mathsf{s}\Big] \\
=\; &\operatorname{Arg}\min_{\mathsf{A}\in\{\mathsf{A}\}} \int_{\{\mathsf{A}\}} \mathsf{A}(\mathsf{s}) \ln\Big(\frac{\mathsf{A}(\mathsf{s})}{\mathsf{D}(\mathsf{s}|\mathsf{v}^{\lambda})}\Big)\mathrm{d}\mathsf{s}.
\end{aligned}$$

The attained bounds are $\gamma_j = \int_{\{\mathsf{s}\}} \mathsf{D}(\mathsf{s}|\mathsf{v}^{\lambda})\mathcal{R}(\mathsf{s}_{j\triangleright}\mathsf{s})\,\mathrm{d}\mathsf{s}$

$$\begin{aligned}
&= -\mathcal{H}(\mathsf{s}_j) - \sum_{s\in\{s\}} \mathsf{s}_j(s) \int_{\{\mathsf{s}\}} \mathsf{D}(\mathsf{s}|\mathsf{v}^{\lambda}) \ln(\mathsf{s}(s))\,\mathrm{d}\mathsf{s} \\
&\underbrace{=}_{(40)} -\mathcal{H}(\mathsf{s}_j) + \psi\Big(\sum_{j\in\{j\}} \lambda_j\Big) - \sum_{s\in\{s\}} \mathsf{s}_j(s)\psi(\mathsf{v}^{\lambda}(s)).
\end{aligned}$$

The wish $(\gamma_j = \mathsf{c}_j\kappa)_{j\in\{j\}}$ gives $|\{j\}|$ equations for λ with $|\{j\}|$ entries

$$\begin{aligned}
&(\mathsf{c}_1 - \mathsf{c}_j)\psi\Big(\sum_{j\in\{j\}} \lambda_j\Big) + \sum_{s\in\{s\}} [\mathsf{c}_j\mathsf{s}_1(s) - \mathsf{c}_1\mathsf{s}_j(s)]\psi(\mathsf{v}^{\lambda}(s)) \\
=\;& \mathsf{c}_1\mathcal{H}(\mathsf{s}_j) - \mathsf{c}_j\mathcal{H}(\mathsf{s}_1),\; j \in \{j\}. \text{ They constrain} \\
& \lambda \in \operatorname{Arg}\min_{\lambda\in\{\lambda\}} [\kappa] = \operatorname{Arg}\min_{\lambda\in\{\lambda\}} [\gamma_1] \\
=\;& \operatorname{Arg}\min_{\lambda\in\{\lambda\}} \Big[\psi\Big(\sum_{j\in\{j\}} \lambda_j\Big) - \sum_{s\in\{s\}} \mathsf{s}_1(s)\psi(\mathsf{v}^{\lambda}(s))\Big] \quad \blacksquare
\end{aligned}$$

6) Discussion of Merging: The search for the merged s that puts together information quantified by pds (34) is motivated here by modelling, Sec. III-B1. It is also needed in knowledge elicitation [8] and serves to a soft cooperation of agents [13], [17], [42]. Within the soft cooperation, the agents offer both environment and ideal slice pds to their neighbours. They use the merged pds for designing their DM rules. The merged pds, and thus the learning and strategy design, take at least partially into account the influence of other neighbours. In this way, each agent reaches a higher DM quality measured by the RE to the unchanged agent's ideal pd. This cooperation way respects limited deliberation resources of the agent. It needs no mediator as the algorithm using Prop. 6 needs none. It is scalable unlike mediator-driven schemes [43], [12], [11].

The solution avoids the incompatibility problem revealed in Sec. III-B2. It exploits all information resources qualified by their creed pd c. Importantly, the creed pd can be gained operationally via Bayes' rule. It is worth stressing that the creed pd is private to the merged-pd-using agent, see Note 12.

The preceding research, represented by [17], is oriented on equalising REs of knowledge sources to the merged pd. It corresponds with the uniform creed pd c but allows any c. The solution given by Prop. 6 avoids the inherent need for information, which can hardly be provided by the cooperating agents. The expectation $\mathcal{E}^{\mathsf{o}}[\mathsf{s}] = \mathsf{s}^{\mathsf{o}}$ of the optimal merger, Prop. 6, may serve as the point estimate of the merged pd[15]

$$\mathsf{s}^{\mathsf{o}} = \sum_{j\in\{j\}} \alpha_j^{\mathsf{o}}\mathsf{s}_j, \quad \alpha_j^{\mathsf{o}} \equiv \frac{\lambda_j^{\mathsf{o}}}{\sum_{j\in\{j\}} \lambda_j^{\mathsf{o}}}. \quad (41)$$

Its form coincides with a linear merging (pooling, [16]), which would use $\alpha^{\mathsf{o}} = \mathsf{c}$. The merging with the weight α^{o} given by Prop. 6 is expected to be better. The weights (41) reflect both the creed pd c *and* the collection $\mathsf{s}^{|\{j\}|}$ of slice pds.

Samples of simulations reflected in Fig. 2 confirms this expectation. In all cases, three slice pds with $|\{s\}| = 18$ and the uniform creed pds were merged according to Prop. 6. Importantly, $\mathsf{A}^{\mathsf{o}} = \mathsf{D}(\mathsf{s}|\mathsf{v}^{\lambda^{\mathsf{o}}})$ assigns to s^{o} (41) its precision

[13] A finite mixture of Dirichlet's pds is generally needed [38] but [39] shows that its use is unnecessary in the inspected context.

[14] A finite amount of behaviours makes the number of slices finite.

[15] The merged pds can be sampled from the optimal merger A^{o}, Prop. 6, which brings the exploration [44] into strategies exploiting it.

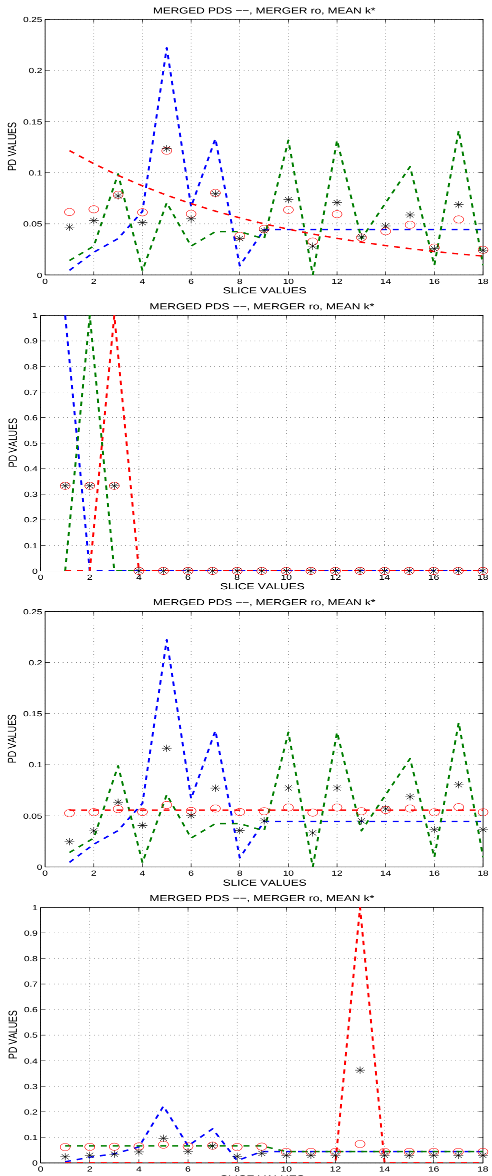

Fig. 2. Merging through Prop. 6. Dashed lines mark slice pds. Red circles mark the merged pd (41). The black stars mark mean of the merged pds enforcing $\alpha^{\mathsf{o}} = \mathsf{c}$. Reading top-down: (1) the figure shows the case with $\alpha^{\mathsf{o}} \approx \mathsf{c}$; (2) the figure confirms that merging becomes Bayes' rule for crisp data; (3) the figure shows the entropy influence: the slice pd with the highest entropy almost coincides with the merged pd; (4) the figure confirms the "conservative nature" of the merger that suppresses the outlying slice probability. ■

via the degrees of freedom $\mathsf{v}^{\lambda^{\mathsf{o}}}$. They strongly influence sensitivity of the merged pd to a further enrichment of the already processed collection $\mathsf{s}^{|\{j\}|}$. The values $\mathsf{v}^{\lambda^{\mathsf{o}}}$ may also control exploration extent, see Note 15. The predecessors of the presented merging lack these properties.

Technically, the used generalised MRE is preferable against the standard one. The latter needs a nontrivial numerical optimisation without a priori clear benefit. This and fact that the standard solution is maximiser of (24) leaves almost no freedom in choosing of the MRE principle version.

The solution in Prop. 6 cleans up that in [39].

7) Ideal Slice Model for Domain Extension: The domain extension (33), Sec. III-B3, depends on the chosen ideal slice model s^{i}. The merged pd s^{o} (41) directly offers to this purpose. Plugin the domain extensions (33) of $\mathsf{s}^{|\{j\}|}$-members into the merged pd s^{o} gives the *implicit optimal merged pd*

$$\mathsf{s}^{\mathsf{o}}(s^{\mathfrak{e}}) = \sum_{j\in\{j\}} \alpha_j^{\mathsf{o}} \frac{\mathsf{s}^{\mathsf{o}}(s_j^{\mathfrak{c}}, \mathsf{s}_j)\mathsf{s}_j(s_j)}{\mathsf{s}^{\mathsf{o}}(s_j)}, \ \forall s^{\mathfrak{e}} \in \{s^{\mathfrak{e}}\}. \tag{42}$$

There, $(\mathsf{s}_j(s_j))_{j\in\{j\}}$ are the given pds of the slices $s_j \in \{s_j\} \subseteq \{s^{\mathfrak{e}}\} = \cup_{j\in\{j\}}\{s_j\}$ with individual complements[16] $s_j^{\mathfrak{c}}$ giving $s^{\mathfrak{e}} = (s_j^{\mathfrak{c}}, s_j)$. The implicit relation (42) was proposed in [13]. Successive approximations offer to its non-unique but always existing solution as analysis and experiments with simpler predecessors shown. The equation (42) waits for its full analysis.

8) Collective Extension: This section applies the MRE principle to a particular case, which is important for knowledge transfer [45] and information fusion [46]. It is an intermediate case of the domain extension, Sec. III-B3, and merging, Sec III-B5. The studied, practically useful, scenario illustrates the potential of the MRE-principle use.

A parametric observation pd $\mathsf{o}_0 \equiv (\mathsf{o}_0(o|p,k))_{o\in\{o\},p\in\{p\}}$ and a prior pd $\mathsf{p}_0 \equiv (\mathsf{p}_0(p|k))_{p\in\{p\}}$ of an unknown parameter $p \in \{p\}$ are given. This knowledge k extends to the knowledge K by a collection of observation models $\mathsf{o}^{|\{j\}|} \equiv (\mathsf{o}_j(o))_{o\in\{o\},j\in\{j\}}$ gained independently of the o_0's structure. The *collective extension* $A = \mathsf{p} = (\mathsf{p}(p|K))_{p\in\{p\}}$ is the opted action. It extends observation models to the joint pds of the observation and parameter $(\mathsf{o}_j\mathsf{p})_{j\in\{j\}\cup\{0\}}$ on the ignorance $G = (o,p) \in \{o\} \times \{p\} = \{G\}$. The unambiguous ignorance model is

$$\mathsf{G}(G|A,K) = \mathsf{G}(o,p|\mathsf{p},\mathsf{o}_0,\mathsf{p}_0,\mathsf{o}^{|\{j\}|},k) = \mathsf{o}_0(o|p,k)\mathsf{p}(p|K) \tag{43}$$

as only the pd $\mathsf{p}_0(o|p,k)$ relates the observation $o \in \{o\}$ to the parameter $p \in \{p\}$ and the action $A = \mathsf{p}$ describes $p \in \{p\}$ by its definition. The ideal ignorance model

$$\mathsf{G}^{\mathsf{i}}(G|A,K) = \mathsf{o}_0(o|p,k)\mathsf{p}_0(p|k) \tag{44}$$

expresses the intention to respect the prior knowledge k given by o_0 and p_0. The collective extensions respecting (27) and Prop. 2 delimit the action set $\{A\} \equiv$

$$\{\mathsf{p}\} \equiv \left\{\mathsf{p}(p|K) : \mathcal{R}(\mathsf{o}_j\mathsf{p}_{\triangleright}\mathsf{o}_0\mathsf{p}_0) \le \gamma_j = \mathsf{c}_j\kappa,\ j \in \{j\}\right\}. \tag{45}$$

The learnable creed pd c is given and the smallest κ guaranteeing $\{\mathsf{p}\} \neq \emptyset$ is chosen, as in Sec. III-B5. DM rules, *collective extensors* $\mathsf{A}(\mathsf{p}|K)$, are constrained by their supports. They have to guarantee the expected version of (45) $\{\mathsf{A}\} \equiv$

$$\left\{\mathsf{A}(\mathsf{p}|K) : \int_{\{\mathsf{p}\}} \mathsf{A}(\mathsf{p}|K)\mathcal{R}(\mathsf{o}_j\mathsf{p}_{\triangleright}\mathsf{o}_0\mathsf{p}_0)\ \mathrm{d}\mathsf{p} \le \gamma_j = \mathsf{c}_j\kappa\right\}. \tag{46}$$

Proposition 7 (Collective Extensions): Let: (a) the creed pd c in (45) be given; (b) LFO 6 be applied to collective extensors $\mathsf{A}^{\mathsf{i}} = \mathsf{A}$; (c) the set $\{\mathsf{p}\} \neq \emptyset$ (45) with the bounds $\gamma_j = \mathsf{c}_j\kappa$ for the smallest κ be considered.

[16]Practically, it is important that when two agents model the same variable the merging is to express it in the same units to make domains of combined slice pds compatible. This comment applies for all cases facing this situation.

Then, the optimal collective extensor deterministically generates the optimal collective extension, cf. EF (16),

$$\begin{aligned} \mathsf{p}^{\mathsf{o}}(p|K) &= \mathsf{p}^{\alpha^{\mathsf{o}}}(p|K) = \frac{\mathsf{p}_0(p|k)\exp\langle\alpha^{\mathsf{o}},\mathsf{w}(p,k)\rangle}{\mathcal{J}(\alpha^{\mathsf{o}})} \\ \langle\alpha^{\mathsf{o}},\mathsf{w}\rangle &= \sum_{j\in\{j\}} \alpha^{\mathsf{o}}_j \mathsf{w}_j \qquad (47) \\ \mathsf{w}_j &\equiv \mathsf{w}_j(p,k) \equiv \int_{\{o\}} \mathsf{o}_j(o)\ln(\mathsf{o}_0(o|p,k))\,\mathrm{d}o. \end{aligned}$$

The optimal non-negative weight α^{o} is

$$\begin{aligned} \alpha^{\mathsf{o}} &\in \mathrm{Arg}\min_{\sum_{j\in\{j\}}\alpha_j\le 1}\Big[-\frac{\partial\mathcal{J}(\alpha)}{\partial\alpha_1}+\lim_{\zeta\to 1}\frac{\partial\ln(\mathcal{J}(\zeta\alpha))}{\partial\zeta}\Big] \\ \text{for}\quad & \mathsf{c}_1\frac{\partial\mathcal{J}(\alpha)}{\partial\alpha_j}-\mathsf{c}_j\frac{\partial\mathcal{J}(\alpha)}{\partial\alpha_1}[\mathsf{c}_j-\mathsf{c}_1]\lim_{\zeta\to 1}\frac{\partial\ln(\mathcal{J}(\zeta\alpha))}{\partial\zeta} \\ &= \mathsf{c}_0\mathcal{H}(\mathsf{o}_j)-\mathsf{c}_j\mathcal{H}(\mathsf{o}_1),\ j\in\{j\}. \end{aligned}$$

Proof The form (47) of $\mathsf{p}^{\alpha}(p|K)$ with $\alpha_j = \frac{\lambda_j}{1+\sum_{j\in\{j\}}\lambda_j} > 0$ to be chosen follows from $\mathsf{A}^{\mathsf{i}} = \mathsf{A}$ and a rearrangement of the Kuhn-Tucker functional with multipliers $\lambda = (\lambda_j \ge 0)_{j\in\{j\}}$ respecting bounds in (46). Some bounds are active and give $\lambda_j > 0$. For $j \in \{j\}$, the RE in (45) gets the form, with $\mathcal{H}(\mathsf{o}_j)$ given by (40) and expressed via the normalising $\mathcal{J}$ in (47),

$$\mathcal{R}(\mathsf{o}_j\mathsf{p}^{\alpha}{}_{\triangleright}\mathsf{o}_0\mathsf{p}_0) = -\mathcal{H}(\mathsf{o}_j) - \frac{\partial\mathcal{J}(\alpha)}{\partial\alpha_j} + \lim_{\zeta\to 1}\frac{\partial\ln(\mathcal{J}(\zeta\alpha))}{\partial\zeta}$$

The wish $(\gamma_j = \mathsf{c}_j\kappa)_{j\in\{j\}}$ for active bounds in (45) gives $|\{j\}|$ equations for α with $|\{j\}|$ entries. The identity $\mathrm{Argmin}_{\alpha}[\kappa] = \mathrm{Argmin}_{\alpha}[\gamma_1]$ closes the proof. ■

9) Discussion of Collective Extensors: The solution makes Bayes' rule applicable to probabilistic information about observations. This fact motivated its detailed inspection [47] and has led to important applications [48], [8]. Unlike its predecessors, the discussed version: (a) jointly handles several predictors; (b) is independent of the interpretation of o and p; (c) "naturally" introduces and optimises the weights α of $\mathsf{w}(p,k)$; (d) exploits the optional, but learnable, creed pd c; (e) avoids an undesirable ambiguity in the merging way [16].

IV. Related Work

The paper [20] we base on exploits FPD to a unified derivation and generalisation of approximation and minimum RE principles. As a first presentation of this novel way it lacks elaborated argumentation, exploitation, discussion and numerical examples. Our paper tries to remove these weak points. The paper [20] builds, among others, on outcomes of the paper [13], which mainly focuses on merging pds formulated as Bayesian estimation of an optimal merger. It uses an ad hoc meta-model and represents reasonable heuristic solutions that cannot serve as prescriptive ones. The adept for a prescriptive merging [39] is exploited in this text.

Other reference samples are mentioned on the fly. Textbooks, author's and his co-workers works dominate as we refer about a systematic process of building a unified solution. We are well aware, appreciate and (often, indirectly) exploit the related research. The next sketch just indicates it.

The paper falls into the research stream addressing deductively dynamic DM. Subjectively, seminal works related to Bayesian DM are [49], [21], [22], to cybernetics [50], [51], to artificial intelligence [52], [53] and to control [54].

The use of RE (KL) is already a part of folklore. Its exploitation in dynamic DM (control) can be tracked to [55]. DM meta-tasks based on it are in [33], [36], [24]. RE-based FPD was proposed in [1], generally elaborated in [2] and axiomatised in [3], [4]. An independently developed KL control [56], [57] or KL-constrained optimisation, [58], [59] and KL regularisation [60] have a significant overlap with FPD. The concept of random utilities [61] also has a lot of common with FPD but its extent and "language" differ.

The addressed problems have been repeatedly solved by variety inductive ways having natural overlaps with the proposed methodology. For instance, differing domains of sources can be formulated as the problem of missing data addressed by many methods, e.g. [62]. Our solution avoids the need to go through their rich collection. The paper [63] indicates how hard such a choice can be.

V. Concluding Remarks

a) Main Achievement: Methodologically, the inevitable inductive parts of DM are addressed as deductive DM meta-tasks solved in the way, which minimizes the extent of extra choice of methods and their optional parameters.

b) Some Open Problems: In re theory: (a) The assumption that the behaviour set is specified beforehand should be relaxed. It seems inevitable for a systematic knowledge transfer and life-long learning [64]. (b) A systematic, broadly accepted, conceptual support of autonomous agents acting in an open environment [65] still does not exist. (c) A deductive (?) inclusion of computational complexity aspects into the problem formulation and solution, which would lead to truly universal artificial intelligence [52], is highly desirable.

In re applications: (a) The indicated optimisations call for algorithmic and software solutions. (b) The creed learning needs a detailed elaboration. (c) Simulation studies are needed and case-based experience is to be accumulated.

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