Approximate Synchronization of Complex Network Consisting of Nodes With Minimum-Phase Zero Dynamics and Uncertainties

BRANISLAV REHÁK[ID] AND VOLODYMYR LYNNYK[ID]

The Czech Academy of Sciences, Institute of Information Theory and Automation, 182 00 Prague, Czech Republic

Corresponding author: Branislav Rehák (rehakb@utia.cas.cz)

ABSTRACT A synchronization algorithm of nonlinear complex networks composed of nonlinear nodes is designed. The main idea is to apply the exact feedback linearization of every node first, then applying methods for synchronization of linear complex networks. The nodes need not admit full exact feedback linearization, however, they are supposed to be minimum-phase systems. To achieve the synchronization of the observable parts of the nodes, an algorithm based on the convex optimization (to be specific, on linear matrix inequalities) is proposed. Then, it is demonstrated that, using the minimum-phase assumption, the non-observable part of the nodes is synchronized as well. The algorithm for synchronization of the observable parts of the nodes can be used to design a control law that is capable of maintaining stability in presence of certain variations of the control gain. Uncertainties in the parameters are also taken into account. Two examples illustrate the control design.

INDEX TERMS Complex networks, nonlinear systems, linear matrix inequalities, robust control.

I. INTRODUCTION

A. STATE OF THE ART

Many natural as well as artificial systems in the real world can be represented as complex networks with a large number of interconnected units [1]. Examples of complex networks include World Wide Web, Internet, food webs, metabolic networks, biological neural networks, collaboration networks, social networks, electric power grids, etc. A complex network may be represented by a graph composed of nodes connected by links. Due to the nature of the different complex networks, the interconnection between nodes have many different topological forms like ring, star, tree, etc. [2], [3].

Synchronization appears to be one of significant types of interconnected systems' collective dynamics. It is a fundamental mechanism in nature that relates to different phenomena in physical, chemical, and biological systems [4]. Extensive research of the identical (complete, full) synchronization of chaotic coupled systems was started by [5]. It was demonstrated that the necessary condition for synchronization is the presence of a spanning tree in the topological graph of the complex network [2], [6], [7]. In the identical

synchronization, the state variables of every system converge towards each other [2], [8].

The problem of synchronization with different kinds of perturbation like irregular communication delays [9], [10], packets dropouts [11], quantization [12], [13], communication link failures [14], nonconstant interconnection topologies [15] are widely discussed by the control engineering community. Synchronization problem is also known as consensus problem in the recent terminology of control theory, see e.g., [9], [16], [17]. A related problem is the problem of handling large data sets in large-scale distributed learning algorithms, see [18].

Mostly the mathematical models of the complex networks demonstrate the nonlinear behavior. Several approaches based on replacing the nonlinearity by some uncertain terms that are subsequently estimated via the Young's inequality were proposed, [19]. An alternative approach to nonlinear systems is the exact feedback linearization, see [20]. There are many examples of implementing this method in the control of the complex systems. Exact feedback linearization-based control of vehicle platoons can be found in [21]. In [22] authors present a control algorithm of identical affine-in-control systems. Adaptive control laws have also been developed; let us mention an adaptive consensus output regulation designed for a strict-feedback form of nonlinear

systems in [23], [24]. The same problem of second-order nonlinear systems is treated in [25]. The consensus of nonlinear nodes using static output feedback is solved in [26]. Synchronization of complex networks with nontrivial zero dynamics, however, without taking uncertainties into account, was presented in [27]. This approach was applied to the synchronization of complex networks with nonlinear nodes with time delays in [28]. Since the Hindmarsh-Rose neuron is a minimum-phase system, the ideas from the aforementioned papers were applied to the design of the synchronization algorithm for a network composed of these neurons in [29]. Let us also mention the problem of multi-agent synchronization with non-constant topologies, studied e.g. in [15] where a resilient consensus is achieved under sampled signals.

The theory of large-scale systems is related to the problem of the synchronization of the complex network. The control algorithms for linear systems, where "every subsystem is connected with every other" (so-called symmetrically interconnected systems), can be found e.g. in [30], even for systems with delays in the control loop. The control of linear interconnected systems with a more general interconnection topology is presented in [31] while the control of nonlinear large-scale systems based on exact feedback linearization was studied in [32]. Some ideas of this paper are adopted here for the problem of the synchronization of complex networks.

A common advantage of the approaches developed in these papers is the independence of the control design complexity of the number of subsystems. To be specific, the control law is designed using a set of linear matrix inequalities (LMIs) so that complexity of this problem (measured as the dimension of the matrices involved) does not depend on the number of subsystems. This algorithm is based on the methods for the robust control of an uncertain linear system which is often solved using LMIs. Recently, the so-called descriptor approach was used for the solution of the robust control problem of linear systems with time delay, see e.g. [33], [34]. However, as pointed out in [35], the performance of this method is superior to the "classical" one, even in the case of delay-free systems. This approach is used in this paper.

In various control tasks, one cannot assume a precise value of the controller parameters due to implementation imprecision, changes in time, etc. Therefore, the non-fragile control was developed. Here, the control gain is designed so that stabilization of the controlled system is guaranteed even in the presence of the control gain variations. They can be caused e.g. by degradation in time, dependencies of the actuators or the controllers on temperature or other parameters of the environment etc. As such phenomena cannot be avoided, the control law must be designed so that it can guarantee the desired performance even in presence of these changes. These perturbations of the control gain can be additive or multiplicative - the latter case is considered in [36], [37] and also in this paper. The non-fragile networked systems control, with both additive as well as multiplicative variations of the control gain, is presented in [38].

B. PURPOSE AND OUTLINE OF THE PAPER

Nonlinear complex networks are often encountered in practice. The purpose of the paper is to find an efficient synchronizing control for these systems. To be specific:

- To present an algorithm for synchronizing a complex network with identical nonlinear nodes based on the exact feedback linearization. As this procedure precisely matches the nonlinear terms in the node's description, the resulting algorithm will have performance superior to those algorithms based on approximative linearization of the node dynamics.
- Networks with nodes that have a nontrivial minimum-phase zero dynamics are investigated, synchronization of this part is also proved. This is important since in practice, many networks are composed of devices described by systems with nontrivial zero dynamics.
- The synchronization is guaranteed even in the presence of perturbations of the control gain (non-fragile control design). Note that, from the practical point of view, these changes may encompass also changes in the actuators.

The results are achieved by combining the exact linearization with robust control methods. It is demonstrated that the zero synchronization error cannot be achieved in the presence of the disturbances. However, the norm of the error can be estimated. We believe this problem has not been studied in this setting so far.

Outline of the paper: in Section II, basic notions from the graph theory are repeated, while Section III describes the application of the exact linearization to the nodes and defines the uncertainties that can occur in the node description. The synchronization of the observable part using robust control methods is presented in the fourth section. The non-fragile controller design guaranteeing synchronization (up to an error due to uncertainties) is described in the fifth section. The sixth section contains proof of the synchronization of the non-observable part. The Example section and conclusions follow. Some technical lemmas are concentrated in the Appendix.

C. NOTATION

1) If P is a symmetric square matrix, then the inequality $P > 0$ means matrix P is positive definite;
2) In matrices, the zero blocks are denoted by 0; dimensions of these blocks will be clear from the context;
3) The symbol I_m denotes the m-dimensional identity matrix; for brevity, N-dimensional identity matrix is denoted by I, for definition of N, see below;
4) The symbol $\mathrm{diag}(\dots)$ denotes a diagonal matrix composed of blocks in parentheses: $\mathrm{diag}(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$;
5) The symbol $L_f(h)$ denotes the following Lie derivative $L_f h(x)$: $L_f h(x) = \nabla h(x).f(x)$.

II. GRAPH THEORY

Let us analyze a complex network which is composed of N identical nodes. Let $f, g : \mathbb{R}^r \to \mathbb{R}^r$, $h : \mathbb{R}^r \to \mathbb{R}$ be

sufficiently smooth functions, $f(0) = 0$, $h(0) = 0$, $g(0) \neq 0$ and N be a positive integer. Then, the ith node is defined by

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \, x_i(0) = x_{i,0},$$
$$y_i = h(x_i), \tag{1}$$

for all $i = 1, \ldots, N$. Further assumptions about these functions are introduced in Section III.

In the sequel, only the most essential facts from the graph theory used for the analysis of complex networks is presented; more details can be found in [17].

The nodes are denoted by integers from the set $\mathbf{N} = \{1, \ldots, N\}$. Let the set $\mathbf{E} \subset \mathbf{N} \times \mathbf{N}$ be defined as follows: $(i, j) \in \mathbf{E}$ if and only if the node i sends information to the node j. It is assumed that $(i, i) \notin \mathbf{E}$. The directed graph (or digraph) describing the topology of the node network is then defined as $\mathbf{G} = (\mathbf{N}, \mathbf{E})$. It contains no loops. An undirected graph is a directed graph satisfying the condition: for every $i, j \in \mathbb{N}$ holds: if $(i, j) \in \mathbf{E}$ then $(j, i) \in \mathbf{E}$.

In the sequel, only undirected graphs will be considered.

For any $i \in \mathbb{N}$ define the set of neighbors of the node i (denoted by $\mathcal{N}_i$) by $\mathcal{N}_i = \{j \in \mathbb{N} \mid (j, i) \in \mathbf{E}\}$.

The $N \times N$-dimensional *adjacency matrix* $J = (e_{ij})$ is defined as $J_{ij} = 1$ if and only if $(i, j) \in \mathbf{E}$, otherwise $J_{ij} = 0$. Let us also define the Laplacian matrix L by $L = \text{diag}(\sum_{j=1}^{N} J_{1j}, \ldots, \sum_{j=1}^{N} J_{Nj}) - J$.

The graph $\mathbf{G}$ is said to *contain a spanning tree* if, for every $i, j \in \mathbf{N}$, there exists a directed path from the node i to j.

The following result can be found in [39]:

Lemma 1: If the undirected graph $\mathbf{G}$ contains a spanning tree, then 0 is a simple eigenvalue of the Laplacian matrix L corresponding to the eigenvector $\mathbf{e} = (1, \ldots, 1)^T \in \mathbb{R}^N$. Moreover, there exist an orthogonal matrix T and a diagonal matrix Δ such that

$$T^T L T = \Delta. \tag{2}$$

Let $\mathbf{e} = (1, \ldots, 1)^T \in \mathbb{R}^N$. Then the following corollary holds:

Corollary 1: Under the assumptions of Lemma 1, one has

$$L\mathbf{e} = 0. \tag{3}$$

Without loss of generality, it is possible to assume that $\Delta = \text{diag}(0, d_1, \ldots, d_{N-1})$ where d_i are constants satisfying $0 < d_1 \leq \cdots \leq d_{N-1}$.

Let $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$. The solution of the synchronization problem means finding a control u_i guaranteeing

$$\lim_{t \to \infty} \sum_{i=1}^{N} \|x_i(t) - \bar{x}(t)\| = 0. \tag{4}$$

Unfortunately, it is difficult to achieve this goal in the presence of uncertainties. Instead, the goal to achieve is defined as follows: we aim to find control signals u_i so that there exist a class-$\mathcal{K}$ function β_1 and a class-$\mathcal{KL}$ function β_2 so that

$$\sum_{i=1}^{N} \|x_i(t) - \bar{x}(t)\| \leq \beta_2(\|x(0)\|, t) + \beta_1(\bar{x}). \tag{5}$$

The most important constraint is that the control signal u_i is computed from the ith node's state and the states of its neighbors

III. EXACT FEEDBACK LINEARIZATION

The details about the exact feedback linearization and definition of the relative degree and zero dynamics are extensively covered by [20].

Assumption 1: System (1) has relative degree $n \leq r$.

Thus, there exists an integer $n \leq r$ satisfying

1) $L_g L_f^{n-1} h(x)(0) \neq 0$,

2) for all $j = 1, \ldots, n - 2$ holds $L_g L_f^j h(x)(0) = 0$.

The ith node is transformed by the nonlinear state transformation $\mathcal{T}: \mathbb{R}^r \to \mathbb{R}^r$ by $\mathcal{T}(x_i) = \xi_i'$ where

$$\xi_{i,1}' = h(x_i), \tag{6}$$
$$\xi_{i,2}' = L_f h(x_i), \tag{7}$$
$$\vdots$$
$$\xi_{i,n}' = L_g L_f^{n-1} h(x_i)u_i - L_f^n h(x_i), \tag{8}$$
$$\xi_{i,m}' = x_{i,m}, \; m = n + 1, \ldots, r. \tag{9}$$

Let us also define vectors ξ'', η''

$$\xi_i'' = (\xi_{i,1}', \ldots, \xi_{i,n}')^T, \quad \eta_i'' = (\xi_{i,n+1}', \ldots, \xi_{i,r}')^T,$$
$$\phi(\xi_i', \eta_i', v_i) = (f_{n+1}(x_i, u_i), \ldots, f_r(x_i, u_i))^T.$$

The vector ξ_i'' is called the *observable part*. The remaining states η_i'' are called *non-observable part*.

In the next step, the following transformation of the control is defined as

$$v_i = L_g L_f^{n-1} h(x_i)u_i - (L_f^n h(x_i) - L_f^n h(x_i)(0)). \tag{10}$$

We define functions $\Psi: \mathbb{R}^n \to \mathbb{R}$, $\Phi_1: \mathbb{R}^n \to \mathbb{R}$, $\Phi: \mathbb{R}^n \to R$ and matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ as

$$\Psi(\xi_i'', \eta_i'') = L_g L_f^{n-1} h(\mathcal{T}^{-1}(\xi_i'')), \tag{11}$$
$$\Phi_1(\xi_i'', \eta_i'') = L_f^n h(\mathcal{T}^{-1}(\xi_i'')), \tag{12}$$
$$\Phi(\xi_i'', \eta_i'') = \Phi_1(\xi_i'', \eta_i'') - \Phi_1(0)\xi_i'', \tag{13}$$

$$A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ & & \Phi_1(0) & \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

This implies that the ith node obeys the equation

$$\dot{\xi}_i' = \begin{pmatrix} \dot{\xi}_i'' \\ \dot{\eta}_i'' \end{pmatrix} = \begin{pmatrix} A\xi'' + Bv_i \\ \phi(\xi'', \eta_i'', v_i). \end{pmatrix} \tag{14}$$

Relation (10) can be expressed as

$$u_i = \frac{1}{\Psi(\xi_i'', \eta_i'')}\left(v_i + (\Phi(\xi_i'', \eta_i'') - \Phi(0)\xi')\right) \tag{15}$$

where function v_i – the control signal in the transformed coordinates – is designed in the sequel. It will be shown that this control signal depends on ξ_i' and ξ_j' for all neighboring nodes.

The control law (15) is implemented. Note, however, that the nonlinear terms in (15) should exactly match the corresponding terms in (8). However, this might not always be the case as this typically requires precise knowledge of

system's parameters. Hence an uncertainty in the function Φ can appear.

In particular, function Φ is decomposed as $\Phi = \Phi_n + \widetilde{\Phi}$ where Φ_n is the "nominal part" - this function is known, it is used to compute the controller etc. On the other hand, $\widetilde{\Phi}$ is the unmodeled dynamics that will be treated as uncertainty. The function Ψ is decomposed analogously as $\Psi = \Psi_n + \widetilde{\Psi}$.

Due to the presence of uncertainties, the control applied to the system is not the one given by (15) but

$$u_{i,n} = \frac{1}{\Psi_n(\xi_i'', \eta_i'')} \left(v_i + (\Phi_n(\xi_i'', \eta_i'') - \Phi_n(0)\xi_i') \right). \quad (16)$$

Then the observable part of the transformed system reads

$$\dot{\xi}_i'' = A\xi_i'' + \Psi(\xi_i'', \eta_i'')u_{i,n} - (\Phi(\xi_i'', \eta_i'') - \Phi(0)\xi_i'). \quad (17)$$

Substituting (16) into this equation yields

$$\dot{\xi}_i'' = A\xi_i'' + B\Big(\frac{\Psi(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')} v_i + \frac{\Psi(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}(\Phi_n(\xi_i'', \eta_i'')$$
$$- (\Phi(\xi_i'', \eta_i'')) - \frac{\Psi(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}\Phi_n(0)\xi_i' - \Phi(0)\xi_i') \Big). \quad (18)$$

Denote $\widetilde{\Psi}(\xi_i'', \eta_i'') = \Psi(\xi_i'', \eta_i'') - \Psi_n(\xi_i'', \eta_i'')$, $\widetilde{\Phi}(\xi_i'', \eta_i'') = \Phi(\xi_i'', \eta_i'') - \Phi_n(\xi_i'', \eta_i'')$. Then (18) reads

$$\dot{\xi}_i'' = A\xi_i'' + Bv_i$$
$$+ B\frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')} v_i + B\Big(\frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}\Phi_n(\xi'', \eta'')$$
$$+ (1 + \frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')})\widetilde{\Phi}(\xi_i'', \eta_i'')$$
$$- \frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}\Phi_n(0)\xi' - (1 + \frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')})\widetilde{\Phi}(0)\xi' \Big). \quad (19)$$

The terms containing $\widetilde{\Psi}$ and $\widetilde{\Phi}$ can be regarded as uncertainty. It is assumed that there exist $n \times n$-dimensional matrices $D_1, E_1, D_2, E_2, D_3, E_3$ and, for every $i = 1, \ldots, N$ measurable matrix-valued functions $F_{1,i}, F_{2,i}, F_{3,i}$ defined on $[0, \infty)$ such that $\|F_{j,i}(t)\| \leq 1$ for all $t \in [0, \infty)$, all $j = 1, 2, 3$ and all $i = 1, \ldots, N$ and, moreover, the following holds:

$$D_1 F_{1,i}(t) E_1 \xi'' + D_3 F_{3,i}(t) E_3 \eta''$$
$$= B\Big(\frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}\Phi_n(\xi'', \eta'')$$
$$+ (1 + \frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')})\widetilde{\Phi}(\xi_i'', \eta_i''),$$
$$- \frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}\Phi_n(0)\xi' - (1 + \frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')})\widetilde{\Phi}(0)\xi' \Big)$$
$$D_2 F_{2,i}(t) E_2 = B\frac{\widetilde{\Psi}(\xi_i'', \eta_i'')}{\Psi_n(\xi_i'', \eta_i'')}.$$

The, Eq. (17) reads

$$\dot{\xi}_i'' = A\xi_i'' + Bv_i + D_1 F_{1,i}(t) E_1 \xi_i''$$
$$+ D_2 F_{2,i}(t) E_2 v_i + D_3 F_{3,i}(t) E_3 \eta_i'' \quad (20)$$

and, if the control of the ith subsystem is $v_i = K\xi_i''$ for some matrix K, the overall system can be written in form

$$\dot{\xi}'' = (I \otimes A)\xi'' + (L \otimes BK)\xi''$$
$$+ (I \otimes D_1)\text{diag}(F_{11}(t), \ldots, F_{1N}(t))(I \otimes E_1)\xi''$$
$$+ (I \otimes D_2)\text{diag}(F_{21}(t), \ldots, F_{2N}(t))(L \otimes E_2 K)\xi''$$
$$+ (I \otimes D_3)\text{diag}(F_{31}(t), \ldots, F_{3N}(t))(I \otimes E_3)\eta''. \quad (21)$$

Remark 1: Denote $\bar{\bar{\xi}} = e \otimes \frac{1}{N}\sum_{i=1}^N \xi_i''$. With (3), one can rewrite Eq. (21) as

$$\dot{\xi}'' = (I \otimes A)\xi'' + (L \otimes BK)(\xi'' - \mathbf{1} \otimes \bar{\bar{\xi}})$$
$$+ (I \otimes D_1)\text{diag}(F_{11}(t), \ldots, F_{1N}(t))(I \otimes E_1)\xi''$$
$$+ (I \otimes D_2)\text{diag}(F_{21}(t), \ldots, F_{2N}(t))(L \otimes E_2 K)$$
$$\times (\xi'' - \mathbf{e} \otimes \bar{\bar{\xi}})$$
$$+ (I \otimes D_3)\text{diag}(F_{31}(t), \ldots, F_{3N}(t))(I \otimes E_3)\eta''. \quad (22)$$

IV. SYNCHRONIZATION IN THE OBSERVABLE PART
A. AVERAGE DYNAMICS
The uncertainties are not supposed, in general, to be equal for all nodes. Hence symmetry in the complex network is violated. This, in turn, is reflected into a steady error whose magnitude depends on the dynamics of the nodes' average.

Due to Corollary 1, matrix L has a simple eigenvalue 0, its corresponding eigenvector is $\mathbf{e}$. Define $M = I - \frac{1}{N}\mathbf{ee}^T$. Then, the vector ξ defined as

$$\xi = (M \otimes I_n)\xi'' \quad (23)$$

is called disagreement dynamics.

For $k = 1, 2, 3$ denote

$$\tilde{F}_k(t) = \frac{1}{N}\mathbf{e} \otimes (F_{k1}(t), \ldots, F_{kN}(t)),$$

$$\bar{F}_k(t) = I \otimes \frac{1}{N}\sum_{j=1}^N F_{kj}(t),$$

$$\hat{F}_k'(t) = \text{diag}(F_{k1}(t), \ldots, F_{kN}(t)),$$

$$\hat{F}_k(t) = \hat{F}_k'(t) - \tilde{F}_k(t).$$

Note that, if $F_{k1}(t) = \cdots = F_{kN}(t)$ for some k, then $\hat{F}_k(t) = 0$.

Using these functions, one can derive differential equations that govern the dynamics of $\bar{\bar{\xi}}$ and ξ. First, let us introduce the following notation.

$$\omega_1 = -\Big(I \otimes D_1 \bar{F}_1(t) E_1 \Big)(\mathbf{e} \otimes \bar{\bar{\xi}}'') - \Big(I \otimes D_2 \bar{F}_2(t)$$
$$\times (L \otimes E_2 K)(\mathbf{e} \otimes \bar{\bar{\xi}}'') - \Big(I \otimes D_3 \bar{F}_3(t) E_3 \Big)(\mathbf{e} \otimes \bar{\eta}'')$$
$$= -\Big(I \otimes D_1 \bar{F}_1(t) E_1 \Big)(\mathbf{e} \otimes \bar{\bar{\xi}}'')$$
$$- \Big(I \otimes D_3 \bar{F}_3(t) E_3 \Big)(\mathbf{e} \otimes \bar{\eta}''),$$

$$\omega_2 = \Big(I \otimes D_1)\hat{F}_1'(t)(I \otimes E_1) \Big)(\mathbf{e} \otimes \bar{\bar{\xi}}'') + \Big(I \otimes D_2)\hat{F}_2'(t)$$
$$\times (L \otimes E_2 K) \Big)\mathbf{e} \otimes \bar{\bar{\xi}}''$$
$$+ \Big(I \otimes D_3)\hat{F}_3'(t)(I \otimes E_3) \Big)(\mathbf{e} \otimes \bar{\eta}'')$$

$$= \left(I \otimes D_1 \hat{F}_1'(t)(I \otimes E_1)\right)(\mathbf{e} \otimes \bar{\xi}'')$$
$$+ \left(I \otimes D_3 \hat{F}_3'(t)(I \otimes E_3)\right)(\mathbf{e} \otimes \bar{\eta}''),$$
$$\omega = \omega_1 + \omega_2.$$

The second and fourth equalities are due to (3).

Remark 2: If $F_{11}(t) = \cdots = F_{1N}(t)$ and $F_{31}(t) = \cdots = F_{3N}(t)$ then $\omega(t) = 0$.

Then, from Eq. (21) can be inferred

$$\dot{\xi}_i = \dot{\xi}_i'' - \mathbf{e} \otimes \dot{\bar{\xi}}'', \tag{24}$$

$$\dot{\bar{\xi}}'' = (I \otimes A + L \otimes BK)\mathbf{e} \otimes \bar{\xi}'' + (I \otimes D_1)\tilde{F}_1(t) \tag{25}$$
$$\times (I \otimes E_1)\xi + (I \otimes D_2)\tilde{F}_2(t)(L \otimes E_2 K)\xi$$
$$+ (I \otimes D_3)\tilde{F}_3(t)(I \otimes E_3)\eta + \left(I \otimes D_1 \bar{F}_1(t)E_1\right)$$
$$\times (\mathbf{e} \otimes \bar{\xi}'') + \left(I \otimes D_2 \bar{F}_2(t)\right)(L \otimes E_2 K)(\mathbf{e} \otimes \bar{\xi}'')$$
$$+ \left(I \otimes D_3 \bar{F}_3(t)E_3\right)(\mathbf{e} \otimes \bar{\eta}''),$$

$$\dot{\xi} = (I \otimes A + L \otimes BK)\xi + (I \otimes D_1)\hat{F}_1(t)(I \otimes E_1)\xi$$
$$+ (I \otimes D_2)\hat{F}_2(t)(L \otimes E_2 K)\xi$$
$$+ (I \otimes D_3)\hat{F}_3(t)(I \otimes E_3)\eta + \omega. \tag{26}$$

Due to the last three terms in (26), one cannot expect full synchronization.

B. CONTROL FOR UNCERTAIN COMPLEX NETWORKS
Let us introduce an LMI problem whose solution can be used to guarantee the approximate synchronization of the original system.

Consider system (26).

Let $\alpha > 0$ be a scalar. Consider matrices $\bar{P}, Q \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{m \times n}$, $\bar{P} = \bar{P}^T$, $\bar{P} > 0$, Q nonsingular, and positive scalars γ_i ($i = 1, \ldots, 5$) and ε. With these matrices, define matrices σ_{ij} ($i, j = 1, 2$) by

$$\sigma_{11} = AQ + BY + Q^T A^T + Y^T B^T + \gamma_1 D_1 D_1^T + \gamma_2 D_2 D_2^T,$$
$$\sigma_{12} = \bar{P} - Q + \varepsilon(Q^T A^T + Y^T B^T),$$
$$\sigma_{22} = \varepsilon(-Q - Q^T + \gamma_3 D_1 D_1^T + \gamma_4 D_2 D_2^T),$$

and matrix Σ by

$$\Sigma(A, B, D_1, E_1, D_2, E_2, D_3, Y, \bar{P}, Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon)$$
$$= \begin{pmatrix}
\sigma_{11} & \sigma_{12} & E^T & Y^T \bar{E}^T & E^T & Y^T \bar{E}^T & D_3 & I_n & 0 & 0 & Q^T \\
* & \sigma_{22} & 0 & 0 & 0 & 0 & 0 & 0 & D_3 & I_n & 0 \\
* & * & -\gamma_1 I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma_2 I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\gamma_3 I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\gamma_4 I_n & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma_5 I_n & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma_5 I_n & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\gamma_5 I_n & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\gamma_5 I_n & 0 \\
* & * & * & * & * & * & * & * & * & * & -\frac{1}{\alpha} I_n
\end{pmatrix}. \tag{27}$$

Lemma 2: Consider system (26) and a scalar $\alpha > 0$. Assume there exist matrices $\bar{P}, Q, Y$ as described above and positive scalars γ_i ($i = 1, \ldots, 5$) and ε so that LMIs

$$0 > \Sigma(A, B, D_1, E_1, D_2, E_2, D_3, d_1 Y, \bar{P}, Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon),$$
$$0 > \Sigma(A, B, D_1, E_1, D_2, E_2, D_3, d_{N-1}Y, \bar{P}, Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon)$$

hold. Let $K = YQ^{-1}$. Then there exist constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ such that, for the derivative of functional

$V_\xi(\xi) = \xi^T(I \otimes \bar{P})\xi$ *along trajectories of* (26) *holds*

$$\dot{V}_\xi \leq -c_1 V_\xi(\xi) + c_2 \|\eta\|^2 + c_3 \|\omega\|^2. \tag{28}$$

For the proof, see Appendix, part B, Lemma 53.

V. NON-FRAGILE CONTROL
The results of the previous section were derived for the uncertainties after applying the exact feedback linearization. Even though the structure of many systems, e.g., in robotics, allows exact feedback linearization so that the uncertain terms appear in the last step only, and therefore the definition of the uncertainties $\tilde{\Phi}$ and $\tilde{\Psi}$ is straightforward. The application of this method on general uncertain systems is connected with problems. However, the above considerations can be applied to the non-fragile synchronization of the complex network.

In many cases, the control gain can also be affected by fluctuations. In this case, it is natural to design the control law so that the control performs adequately even in the presence of these fluctuations. In the case of complex networks, the "nominal" control gain is equal for all nodes. However, the fluctuations differ.

The above considerations lead to the design of the so-called non-fragile control. For the ith node, the control v_i is defined as

$$v_i = \left((I_n + \Delta_{K,i}(t))K\right) \sum_{j=1}^{N} l_{ij} \xi_j'' \tag{29}$$

where K is the control gain to be defined, $\Delta_{K,i}(t)$ is its multiplicative perturbation. It is assumed that the perturbations can be expressed as

$$\Delta_{K,i} = D_K F_{K,i}(t)E_K, \tag{30}$$

where D_K, E_K are known matrices of appropriate dimension, equal for every node, and $F_{K,i}(t)$ are measurable matrix-valued functions so that $\|F_{K,i}\| \leq 1$.

For the simplicity, we assume $\tilde{\Phi} = 0$ as well as $\tilde{\Psi} = 0$. Then

$$\dot{\xi}'' = A\xi'' + B(I_n + D_K F_{K,i}(t)E_K)K \sum_{j=1}^{N} l_{ij}\xi_j'' \tag{31}$$

hence the dynamics of the complex network obeys the following equation

$$\dot{\xi}'' = \Big(I \otimes A + L \otimes BK + (I_N \otimes BD_K)$$
$$\times \text{diag}(F_{K,1}(t), \ldots, F_{K,N}(t))(L \otimes E_K K)\Big)\xi'' \tag{32}$$

and, finally, it has the same form as (22) where $D_1 = 0$, $E_1 = 0$, $D_2 = D_K$, $E_2 = E_K$, $D_3 = 0$, $E_3 = 0$, $F_{1,i}(t) = 0$, $F_{2,i}(t) = F_{K,i}(t)$, $F_{3,i}(t) = 0$ for all $i = 1, \ldots, N$ and all $t \geq 0$.

If we define $\bar{F}_K(t) = \frac{1}{N} \sum_{j=1}^{N} F_{K,j}(t)$,
$\tilde{F}_K(t) = \frac{1}{N}\mathbf{e} \otimes (F_{K,1}(t), \ldots, F_{K,N}(t))$,
$\hat{F}_{K,k}(t) = \text{diag}(F_{K,1}(t), \ldots, F_{K,N}(t)) - \tilde{F}_K(t)$.
Then $\omega = -\Big(I \otimes D_2)\hat{F}_2'(t)(L \otimes E_2 K)\Big)\bar{\xi}''$. Note that, if all variations of K are equal ($F_{K,1}(t) = \cdots = F_{K,N}(t)$ for all $i = 1, \ldots, N$ and all $t \geq 0$), then $\omega = 0$.

VI. CONVERGENCE IN THE NON-OBSERVABLE PART

The synchronization of the observable part has been studied so far. This section demonstrates that synchronization is also achieved for the non-observable parts of the nodes.

First, note that the zero dynamics reads

$$\dot{\eta} = \phi(0, \eta, 0) \tag{33}$$

with a continuous function ϕ satisfying $\phi(0, 0, 0) = 0$. As assumed, nodes are exponentially minimum-phase systems, thus (33) is exponentially stable. Denote $U = \frac{\partial \phi}{\partial \eta}(0, 0, 0)$; for the triple (ζ, η, v) define function $\Delta_\eta(\zeta, \eta, v)$ by $\Delta_\eta(\zeta, \eta, v) = \frac{\partial \phi}{\partial \eta}(\zeta, \eta, v) - U$. Since $\phi(0, 0, 0) = 0$, one has $\Delta_\eta(0, 0, 0) = 0$. Moreover, for any $a > 0$ there exists a neighborhood $\mathcal{U}_a$ of the origin enjoying the following property: for any $(\zeta, \eta, v) \in \mathcal{U}_a$ holds $\|\Delta_\eta(\zeta, \eta, v)\| \le a$. Define also $\bar{\eta} = \frac{1}{N} \sum_{i=1}^{N} \eta_i$.

Lemma 3: Assume matrix $R > 0$ and scalar $\alpha_\eta > 0$ satisfy

$$RU + U^T R = -\alpha_\eta I, \tag{34}$$

Let a positive scalar μ satisfy $\mu \|P\| \in (0, \alpha_\eta)$ and let the relation

$$\|\Delta_\eta(\zeta_k(t), \eta_k(t), v_k(t))\| \le \mu \tag{35}$$

hold for all $t > 0$ and $k \in \{1, \dots, N\}$. Moreover, assume existence of a scalar $\kappa > 0$ such that for all $(\zeta', \eta, v') \in \mathcal{U}_\mu$, $(\zeta'', \eta, v'') \in \mathcal{U}_\mu$:

$$\|\phi(\zeta', \eta, v') - \phi(\zeta'', \eta, v'')\| \le \kappa(\|\zeta' - \zeta''\| + \|v' - v''\|). \tag{36}$$

Then, there exist constants $c_1' > 0$, $c_2' > 0$ such that, for the functional V_η defined by $V_\eta = \sum_{i=1}^{N} (\eta_i - \bar{\eta})^T R(\eta_i - \bar{\eta})$ holds

$$\dot{V}_\eta \le -c_1' V_\eta(\eta) + c_2' V_\xi(\xi). \tag{37}$$

Proof: For the derivative of V_η holds

$$\dot{V}_\eta = \sum_{i=1}^{N} (\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \eta_i, v_i) - \phi(\bar{\zeta}, \bar{\eta}, \bar{v})\Big)$$

$$= \sum_{i=1}^{N} (\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \eta_i, v_i) - \phi(\zeta_i, \bar{\eta}, v_i)\Big)$$

$$+ (\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \bar{\eta}, v_i) - \phi(\bar{\zeta}, \bar{\eta}, \bar{v})\Big). \tag{38}$$

First, an estimate of the term $(\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \eta_i, v_i) - \phi(\zeta_i, \bar{\eta}, v_i)\Big)$ is derived. To proceed, note that

$$(\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \eta_i, v_i) - \phi(\zeta_i, \bar{\eta}, v_i)\Big)$$

$$= (\eta_i - \bar{\eta})^T R \frac{\partial \phi}{\partial \eta}(\tilde{\zeta}, \tilde{\eta}, \tilde{v})(\eta_i - \bar{\eta}) \tag{39}$$

with some (unknown) values $\tilde{\zeta}_i, \tilde{\eta}_i, \tilde{v}_i$. Due to the assumption,

$$\frac{\partial \phi}{\partial \eta}(\tilde{\zeta}_i, \tilde{\eta}_i, \tilde{v}_i)(\eta_i - \bar{\eta}) = U(\eta_i - \bar{\eta}) + \Delta_\eta(\tilde{\zeta}, \tilde{\eta}, \tilde{v})(\eta_i - \bar{\eta})$$

and thus, thanks to (34) and (35), one has

$$(\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \eta_i, v_i) - \phi(\zeta_i, \bar{\eta}, v_i)\Big) \le -(\alpha_\eta - \mu)\|\eta_i - \bar{\eta}\|^2.$$

To estimate the last term in (38), the Young's inequality is used. With $\beta > 0$ holds

$$|(\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \bar{\eta}, v_i) - \phi(\bar{\zeta}, \bar{\eta}, \bar{v})\Big)|$$

$$\le \beta \|\eta_i - \bar{\eta}\|^2 + \frac{1}{\beta}\Big(\phi(\zeta_i, \bar{\eta}, v_i) - \phi(\bar{\zeta}, \bar{\eta}, \bar{v})\Big)^2. \tag{40}$$

The multivariate mean value theorem yields that (36) attains the form

$$|(\eta_i - \bar{\eta})^T R\Big(\phi(\zeta_i, \bar{\eta}, v_i) - \phi(\bar{\zeta}, \bar{\eta}, \bar{v})\Big)|$$

$$\le \beta \|\eta_i - \bar{\eta}\|^2 + \frac{\kappa^2}{\beta}\Big(\|\zeta_i - \bar{\zeta}\| + \|v_i - \bar{v}\|\Big)^2. \tag{41}$$

Summarizing the previous results, (39) and (41) imply

$$\dot{V}_\eta \le -(\alpha_\eta - \beta - \mu)$$

$$\times \sum_{i=1}^{N} \Big(\|\eta_i - \bar{\eta}\|^2 + \frac{\kappa^2}{\beta}(\|\zeta_i - \bar{\zeta}\| + \|v_i - \bar{v}\|)^2\Big) \tag{42}$$

and, consequently, there exist constants $c_1' > 0$, $c_2' > 0$ such that (37) holds. $\square$

Theorem 1: Let assumptions of Lemma 3 hold, moreover, let $c_1 > c_2'$, $c_1' > \bar{c}_2'$. Then there exist a class-$\mathcal{K}$ function β_1 and a class-$\mathcal{KL}$ function β_2 such that the following inequality holds

$$\|\xi\| + \|\eta\| \le \beta_2(\|\xi(0)\| + \|\eta(0)\|, t) + \beta_1(\|\omega\|). \tag{43}$$

Proof: Moreover, from (28) follows that there exists a constant $\bar{c}_2' > 0$ such that

$$\dot{V}_\xi \le -c_1 V_\xi(\xi) + \bar{c}_2' V_\eta(\eta) + \bar{c}_3 \|\omega\|^2. \tag{44}$$

The sum of inequalities (44) and (37) yields

$$\dot{V}_\xi + \dot{V}_\eta \le -(c_1 - c_2') V_\xi(\xi) - (c_1' - \bar{c}_2') V_\eta(\eta) + \bar{c}_3 \|\omega\|^2. \tag{45}$$

As assumed, $\min(c_1 - c_2', c_1' - \bar{c}_2') > 0$. As a consequence, there exist a class-$\mathcal{K}$ function β_1 and a class-$\mathcal{KL}$ function β_2 so that (43) holds. $\square$

As shown in the following corollary, the case of non-fragile control in the absence of further uncertainties leads to (exact) synchronization of the complex network.

Corollary 2: If $D_1 = 0$, $E_1 = 0$, $D_3 = 0$, $E_3 = 0$ and $\hat{F}_2(t) = 0$ for all $t \ge 0$ then the complex network is synchronized.

Proof: In this case, $\omega = 0$, hence $\dot{V}_\xi + \dot{V}_\eta \le -(c_1 - c_2') V_\xi(\xi) - (c_1' - \bar{c}_2') V_\eta(\eta)$, thus $\eta \to 0$ as well as $\xi \to 0$. $\square$

From the definition of the exact feedback linearization follows that there exists a matrix T such that $(\xi''^T, \eta''^T)^T = T\xi'$. Then $\omega = (I \otimes D_1)(\hat{F}_1'(t) - \bar{F}_1'(t))(I \otimes E_1)(\mathbf{e} \otimes (T\xi')) = (I \otimes D_1)(\hat{F}_1'(t) - \bar{F}_1'(t))(I \otimes E_1)(\mathbf{e} \otimes (T\mathcal{T}(x)))$.

Theorem 2: Let assumptions of Theorem 1 hold. Then there exists a pair of functions β_1', β_2' so that β_1' is a class-$\mathcal{K}$ function, β_2' is a class-$\mathcal{KL}$ function and for all $i = 1, \dots, N$ holds

$$\|x_i - \bar{x}\| \le \beta_2'(\|x(0)\|, t) + \beta_1'(\|\omega'\|). \tag{46}$$

Proof: Since transformation $\mathcal{T}$ is a diffeomorphism, inequality (46) is a direct consequence of inequality (43). $\square$
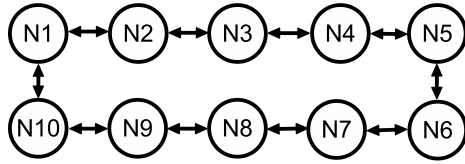
FIGURE 1. Ring topology of the directed complex network consist of ten nodes. Example 1.

VII. EXAMPLES

A. EXAMPLE 1 - NON FRAGILE CONTROL

The non-fragile control synchronizes the following complex network composed of 10 nodes. These equations describe each node ($i = 1, \ldots, 10$)

$$\dot{x}_{i,1} = (\sin^2 x_{i,1} + x_{i,2})$$
$$\dot{x}_{i,2} = 1 - \cos x_{i,1} + u_i,$$
$$\dot{x}_{i,3} = -\sin x_{i,3} + x_{i,1},$$
$$y = x_1.$$

The interconnection of nodes is depicted in Figure 1. Thus the Laplacian matrix has maximal eigenvalue $\lambda_M = 4$ and the minimal nonzero eigenvalue $\lambda_m = 0.382$.

The feedback linearization of each node in the nominal case yields

$$\dot{\xi}''_{i,1} = \xi''_{i,2},$$
$$\dot{\xi}''_{i,2} = v_i,$$
$$\dot{\eta}''_i = -\sin \eta''_i + \xi''_{i,1}.$$

The zero dynamics is $\dot{\eta} = -\sin \eta$, hence it is asymptotically stable around the origin. The system is also a minimum-phase system. Moreover, the observable part of the nominal system is $\dot{\xi}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$.

To define the multiplicative uncertainty $\Delta_{K,i}$, assume existence of measurable real-valued functions $\delta_i : [0, \infty) \rightarrow [-1; 1]$.

The goal is to find matrix K with the following property: if the control of the ith node attains the form $v_i = \sum_{j=1}^N J_{ij} K (1 + 0.1\delta_i(t))(\xi'_j - \xi'_i)$ for j denoting the neighboring nodes of i, then the approximate synchronization of the complex network is achieved in the presence of any multiplicative uncertainty given in terms of the function δ.

Using the inverse transformation, one obtains that, in the nominal case,

$$u_i = K \sum_j \left((x_{j,1} - x_{i,1}, \ \sin^2 x_{j,1} + x_{j,2}) \right.$$
$$\left. - \sin^2 x_{i,1} - x_{i,2}) \right)^T - 2 \sin x_{i,1} \cos x_{i,1} - (1 - \cos x_{i,1}).$$

However, applying this control to the perturbed system yields a discrepancy: the closed-loop reads

$$\dot{x}_{i,1} = (\sin^2 x_{i,1} + x_{i,2}),$$
$$\dot{x}_{i,2} = \delta_i(t)(2 \sin x_{i,1} \cos x_{i,1} + 1 - \cos x_{i,1})$$
$$\quad + (1 + \delta_i(t))K \sum_j ((x_{j,1}, \sin^2 x_{j,1} + x_{j,2}),$$
$$\dot{x}_{i,3} = -\sin x_{i,3} + x_{i,1}.$$

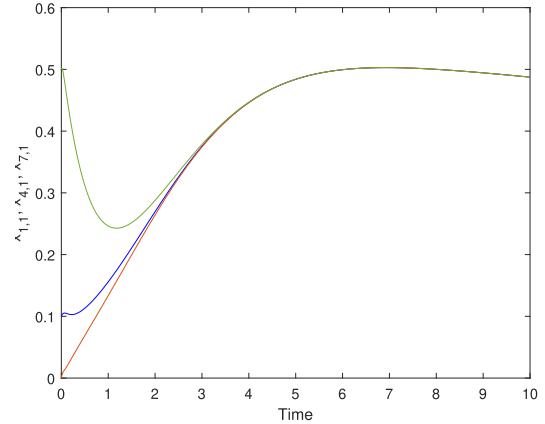**FIGURE 2. States $x_{1,1}$, $x_{4,1}$ and $x_{7,1}$.**

In the transformed coordinates, the observable part obeys the equation

$$\dot{\xi}''_{i,1} = \xi''_{i,2},$$
$$\dot{\xi}''_{i,2} = \sum_j K(\xi''_j - \xi''_i) + \delta_i(t) \sum_j K(\xi''_j - \xi''_i)$$
$$\quad + \delta(t)(2 \sin \xi''_{i,1} \cos \xi''_{i,1} + 1 - \cos \xi''_{i,1}).$$

Note that $|2 \sin s \cos s + 1 - \cos s| \leq 3s$. If one chooses matrices D, E as $D = (0, 1)^T$, $E = (0.3, 0)$ and $F_i(t) = \delta(t)(2 \sin \xi'_{i,1} \cos \xi'_{i,1} + 1 - \cos \xi'_{i,1})$, one has, with the above bounds on the uncertainty δ_i, $\|F_i(t)\| \leq 1$. Similarly, choosing $D_k = (0, 2)^T$ and $E_k = 0.2$ and $F_{k,i}(t) = \delta_i(t)$, one has $\|(L \otimes I)F_k(t)\| \leq 1$.

The algorithm from Sec. IV yields

$$K = (-20.9456, -16.7688).$$

The system was simulated, the control gain was perturbed - it was multiplied with a random signal with uniform distribution in the interval $[-0.1, 0.1]$.

Fig. 2 shows the state $x_{1,1}$ (the blue line), $x_{4,1}$ (the red line) and $x_{7,1}$ (the green line). For the sake of clarity of the figure, the state of the remaining nodes were not plotted here. Moreover, Fig. 3 and Fig. 4 illustrates the states $x_{1,2}$, $x_{4,2}$ and $x_{7,2}$ and $x_{1,3}$, $x_{4,3}$ and $x_{7,3}$. The meaning of the line colors is as in Fig. 2. Finally, the norm of the synchronization error (the norm of the disagreement vector) is depicted in Fig. 5. In this simulations, we can see that this norm decreases in time. This, in turn, means the behavior of all nodes of the complex network is identical. To sum up, despite the perturbations in the control gain, the system is synchronized.

B. EXAMPLE 2 - ROBUST CONTROL OF A NETWORK OF UNDERACTUATED SYSTEMS

The network of 6 interconnected underactuated systems - each of them is a pendulum on a cart - is studied here. The system is thoroughly described in [40]. Hence the description is kept relatively brief here.

The interconnection of the nodes is shown in Fig. 6.

After application of the exact feedback linearization, we can see that the ith node is governed by the following
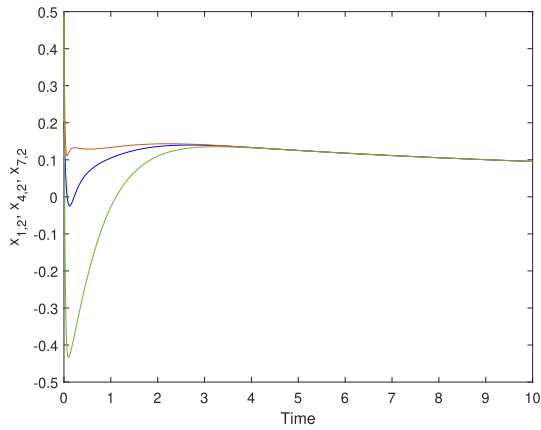
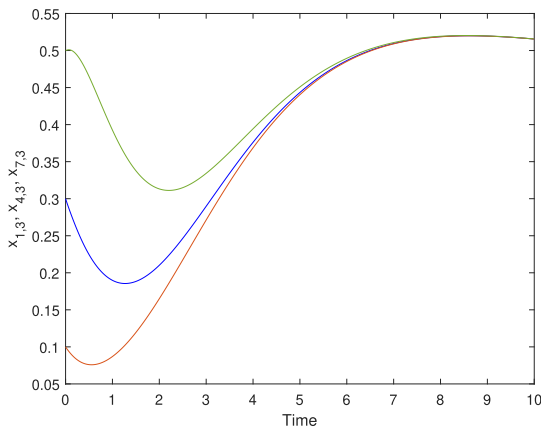**FIGURE 3.** States $x_{1,2}$, $x_{4,2}$ and $x_{7,2}$.

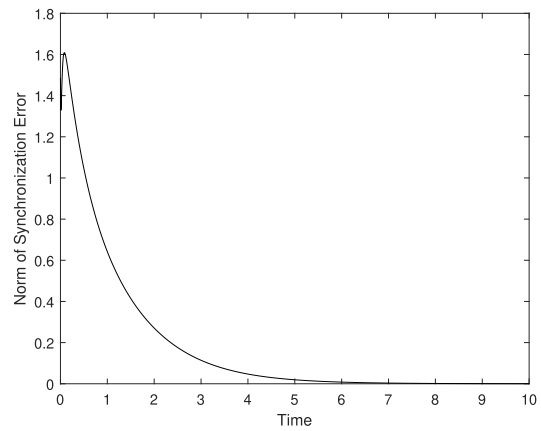**FIGURE 4.** States $x_{1,3}$, $x_{4,3}$ and $x_{7,3}$.
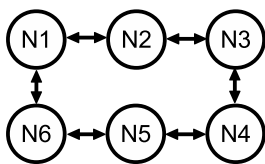

FIGURE 5. Norm of the Synchronization Error.

FIGURE 6. Ring topology of the directed complex network consisting of six nodes. Example 2.
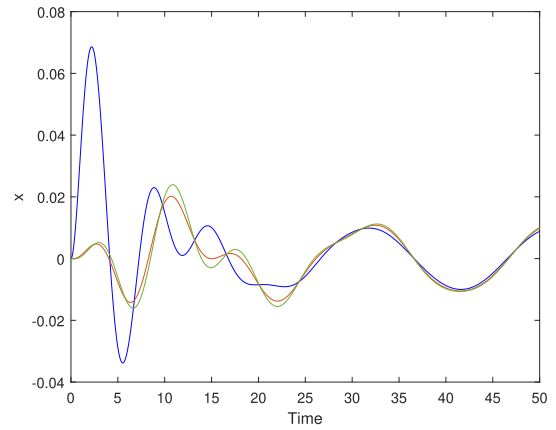
equations

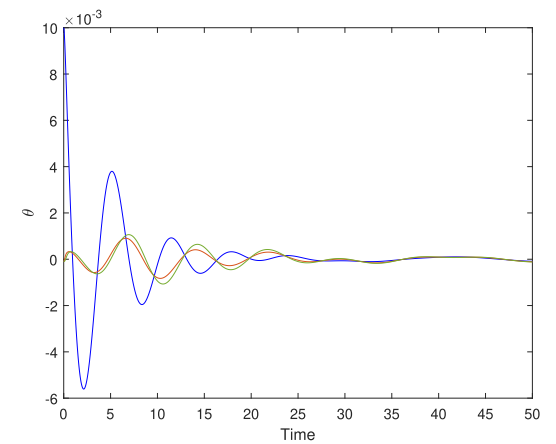$$\ddot{x}_i = u_i, \tag{47}$$

FIGURE 7. States x_1, x_3 and x_5.

FIGURE 8. States θ_1, θ_3 and θ_5.
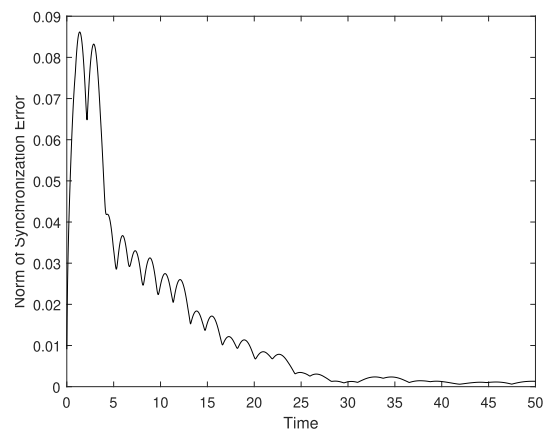
FIGURE 9. Norm of the synchronization error.

$$\ddot{\theta}_i = \frac{g}{l} \sin\theta_i - \frac{1}{l} \cos\theta u_i \tag{48}$$

where x_i is the position of the cart of the ith node, θ_i is the angle of the pendulum, l is the length of the pendulum, g is the gravity acceleration (these parameters are equal for all nodes), and u_i is the control input.

To find the control, define the output as

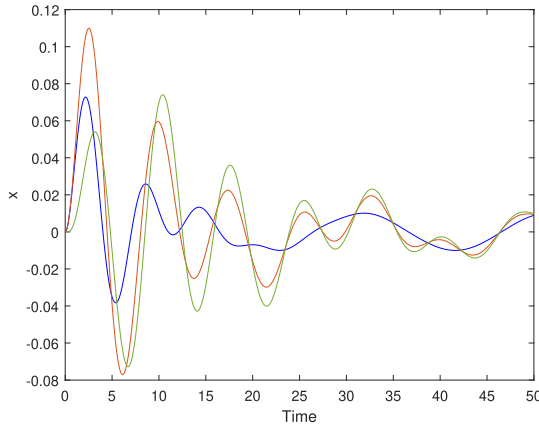$$y_i = x_i + k_1 \sin\theta_i \tag{49}$$
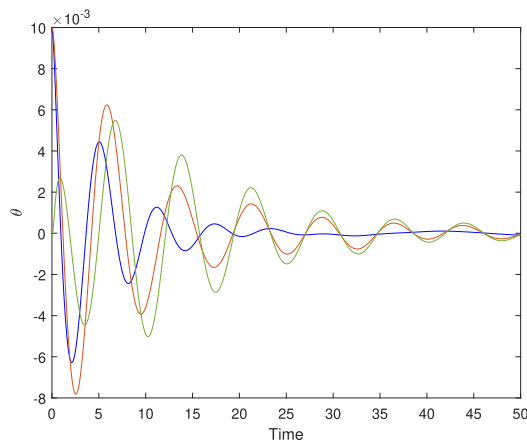
FIGURE 10. States x_1, x_3 and x_5.

FIGURE 11. States θ_1, θ_3 and θ_5.
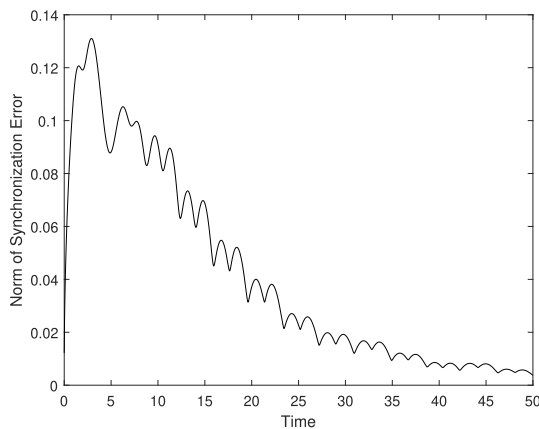

FIGURE 12. Norm of the synchronization error.

with $k_1 = 2$ in our example (this parameter being identical for all nodes). To achieve the minimum-phase property, [40] shows that the term

$$u_i = k_2 \frac{\dot{\theta}_i \cos \theta_i}{\frac{k_1}{l} \cos^2 \theta_i - 1} \qquad (50)$$

can be added, in our case, $k_2 = 20$ was chosen (again, this parameter is equal for all nodes). The relation between the control signal u_i that is fed into the system and the

transformed control input v_i is

$$u_i = \frac{1}{\frac{k_1}{l} \cos^2 \theta_i - 1}$$
$$\times \left(v_i + k_1 \sin \theta_i \left(\frac{g}{l} \cos \theta_i - \dot{\theta}_i^2 \right) + k_2 \dot{\theta}_i \cos \theta_i \right). \qquad (51)$$

The signal v_i is obtained by

$$v_i = k_3 (\dot{x}_i + k_1 \dot{\theta}_i \cos \theta_i + k_2 \sin \theta_i)$$
$$+ k_4 (x_i + k_1 \sin \theta_i + k_2 \int_0^t \sin \theta_i(s) ds.$$

The algorithm presented here gives $k_3 = 7.27$, $k_4 = 13.07$. This control enables to stabilize the vertical position of the pendulum and the position of the cart.

In the numerical simulations, a sinusoidal signal acting as a disturbance was added to the first node.

Figures 7 and 8 show the behavior of the network if all nodes are equal. In Fig. 7, the blue, red, and green lines illustrate the position of the cart of the first, third, and fifth nodes, respectively. Analogous meaning of the line colors is used in Fig. 8 where the angle of the pendulum θ is shown. The norm of the synchronization error in the entire network is depicted in Fig. 9.

Another set of experiments was conducted with a network consisting of 6 nodes where the first, third, and sixth nodes were perturbed - their length was reduced to one-half. The results are shown in Figs. 10-12, showing the position of the cart, the angle of the pendulum, and the overall synchronization error of the first, third and fifth node again. The synchronizing control designed using the robust control tools is still capable of achieving synchronization. However, one can see that the error is significantly larger here.

VIII. CONCLUSION

An algorithm for synchronization of a complex network with nonlinear identical nodes was derived. The proposed control law is robust against uncertainties in the nodes and non-fragile - it can tolerate certain changes of the control gain. The algorithm for the design of this control can be separated into two parts: the exact feedback linearization of the nodes with subsequent design of a robust control for the linearized system. If the nodes are not admitting the full exact feedback linearization but are minimum-phase systems, the proposed approach is also applicable to these systems, yielding synchronization of all states of all nodes. Two examples illustrate the results.

APPENDIX A
DESCRIPTOR APPROACH FOR UNCERTAIN SYSTEMS

This part investigates properties of the descriptor approach based control design for uncertain systems. The structure of the auxiliary system is tailored to fit the structure of the complex network investigated in the previous part of the paper. Consider the ν-dimensional system

$$\dot{x}' = (A' + B'K + D'F'(t)E' + \bar{D}'\bar{F}'(t)\bar{E}'K')x$$
$$+ \bar{\bar{D}}'\bar{\bar{F}}'(t)\bar{\bar{E}}'w' + w'', \ x'(0) = x_0'. \qquad (52)$$

where F', $\bar{F}'$, $\bar{\bar{F}}'$ are measurable matrix-valued functions defined on $[0, \infty)$ such that for all $t \in [0, \infty)$ holds $\|F'(t)\| \leq 1$, $\|\bar{F}'(t)\| \leq 1$, $\|\bar{\bar{F}}'(t)\| \leq 1$ and w', w'' are disturbances.

In the following lemma, it is demonstrated how negative definiteness of matrix Σ defined in (27) allows finding bounds on the solution in the presence of disturbances.

Lemma 4: Consider system (52), let $\alpha > 0$ be given. Assume there exist matrices $\bar{P} = \bar{P}^T, \bar{P} > 0$, Q nonsingular and Y and positive scalars γ_j, $j = 1, \ldots, 5$ and ε such that $\Sigma(A', B', D', E', \bar{D}', \bar{E}', \bar{\bar{D}}', Y, \bar{P}, Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon) < 0$. Then, if $V = x'^T \bar{P} x'$, one has $\dot{V} \leq -\alpha \|x'\|^2 + \gamma_5 (1 + \varepsilon)(w'^T \bar{\bar{E}}'^T \bar{\bar{E}}' w' + w''^T w'')$.

Proof: First, define matrix Ω as

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ * & \omega_{22} \end{pmatrix}$$

with

$$\begin{aligned}
\omega_{11} &= W^T(A' + B'K') + (A' + B'K')^T W \\
&\quad + \gamma_1 W^T D'D'^T W + \frac{1}{\gamma_1} E'^T E' + \gamma_2 W^T \bar{D}' \bar{D}'^T W \\
&\quad + \frac{1}{\gamma_2} K'^T \bar{E}'^T \bar{E}' K' + \frac{\varepsilon}{\gamma_3} E'^T E' + \frac{\varepsilon}{\gamma_4} K'^T \bar{E}'^T \bar{E}' K' \\
&\quad + \frac{1}{\gamma_5} W^T \bar{\bar{D}}' \bar{\bar{D}}'^T W + \frac{1}{\gamma_5} W^T W + \alpha I_n,
\end{aligned}$$

$$\omega_{12} = P - W^T + \varepsilon(A' + B'K')^T W,$$

$$\begin{aligned}
\omega_{22} &= \varepsilon(-W - W^T + \gamma_3 W^T D'D'^T W + \gamma_4 W^T \bar{D}' \bar{D}'^T W \\
&\quad + \frac{1}{\gamma_5} W^T \bar{\bar{D}}' \bar{\bar{D}}'^T W + \frac{1}{\gamma_5} W^T W).
\end{aligned}$$

The descriptor approach is used. This yields, using the Young's inequality, for positive scalars γ_i

$$\begin{aligned}
\dot{V} &= 2\dot{x}'^T P x' + (x'^T W^T + \varepsilon \dot{x}'^T W^T) \\
&\quad \times (-\dot{x}' + (A' + B'K' + D'F'(t)E' + \bar{D}' \bar{F}'(t)\bar{E}' K')x \\
&\quad + \bar{\bar{D}}' \bar{\bar{F}}'(t)\bar{\bar{E}}' w' + w'') \leq 2\dot{x}'^T P x' \\
&\quad + (x'^T W^T + \varepsilon \dot{x}'^T W^T)(-\dot{x}' + (A' + B'K')x') \\
&\quad + x'^T(\gamma_1 W^T D'D'^T W + \frac{1}{\gamma_1} E'^T E' + \gamma_2 W^T \bar{D}' \bar{D}'^T W \\
&\quad + \frac{1}{\gamma_2} K'^T \bar{E}'^T \bar{E}' K' + \frac{\varepsilon}{\gamma_3} E'^T E' + \frac{\varepsilon}{\gamma_4} K'^T \bar{E}'^T \bar{E}' K' \\
&\quad + \frac{1}{\gamma_5} W^T \bar{\bar{D}}' \bar{\bar{D}}'^T W + W^T W)x' + \varepsilon \dot{x}'^T(\gamma_3 W^T D'D'^T W \\
&\quad + \gamma_4 W^T \bar{D}' \bar{D}'^T W + \frac{1}{\gamma_5} W^T \bar{\bar{D}}' \bar{\bar{D}}'^T W + W^T W)\dot{x} \\
&\quad + \gamma_5(1+\varepsilon)(w'^T \bar{\bar{E}}'^T \bar{\bar{E}}' w' + w''^T w'') \leq (x'^T, \dot{x}'^T) \Omega \begin{pmatrix} x' \\ \dot{x}' \end{pmatrix} \\
&\quad - \alpha \|x'\|^2 + \gamma_5(1+\varepsilon)(w'^T \bar{\bar{E}}'^T \bar{\bar{E}}' w' + w''^T w'').
\end{aligned}$$

Since matrix W is assumed to be nonsingular, one can define matries $Q = W^{-1}$, $Y = K'Q$, $\bar{P} = Q^T P Q$ and $\Sigma' = \text{diag}(Q^T, Q^T) \Omega \text{diag}(Q, Q)$. Apparently, $\Omega < 0$ if and only if $\Sigma' < 0$. Then, applying Schur complement seven times, one yields matrix $\Sigma(A', B', D', E', \bar{D}', \bar{E}', \bar{\bar{D}}', Y, \bar{P}, Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon)$. Due to the properties of the Schur complement, $\Sigma(A', B', D',$

$E', \bar{D}', \bar{E}', \bar{\bar{D}}', Y, \bar{P}, Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon) < 0$ if and only if $\Sigma' < 0$. This completes the proof. $\qquad \square$

APPENDIX B
PRACTICAL SYNCHRONIZATION OF A COMPLEX NETWORK SYSTEM COMPOSED OF IDENTICAL NODES

Consider the disagreement dynamics (26).

To state the final result of the appendix, the following notation will be helpful.

$$\begin{aligned}
\Sigma_i &= \Sigma(A, B, D_1, E_1, D_2, \bar{E}, D_3, d_i Y, \bar{P}, Q, \\
&\qquad \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon), \\
\Sigma_\Delta &= \Sigma(I \otimes A, I \otimes B, I \otimes D_1, I \otimes E_1, I \otimes D_2, I \otimes \bar{E}, \\
&\qquad I \otimes D_3, \Delta \otimes Y, I \otimes \bar{P}, I \otimes Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon), \\
\Sigma_L &= \Sigma(I \otimes A, I \otimes B, I \otimes D_1, I \otimes E_1, I \otimes D_2, I \otimes \bar{E}, \\
&\qquad I \otimes D_3, L \otimes Y, I \otimes \bar{P}, I \otimes Q, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \varepsilon),
\end{aligned}$$

Lemma 5: Let there exist matrices $\bar{P} > 0$, Q nonsingular and Y and positive scalars α, γ_j, $j = 1, \ldots, 5$ and ε so that $\Sigma_1 < 0$ and $\Sigma_{N-1} < 0$. Then, with $K = YQ^{-1}$, there exist constants $q_1 > 0$, $q_2 > 0$ such that

$$\dot{V}(\xi) \leq -\alpha \|\xi\|^2 + q_1 \|\eta\|^2 + q_2 \|\omega\|^2. \tag{53}$$

Proof: Obviously, LMIs $\Sigma_1 < 0$ and $\Sigma_{N-1} < 0$ imply $\Sigma_k < 0$ for all $k = 1, \ldots, N - 1$. Let also $d_N = 0$. Then, there exists a permutation matrix Π such that

$$\Sigma_D = \Pi^T \text{diag}(\Sigma_1, \ldots, \Sigma_{N-1})\Pi.$$

Clearly, this implies $\Sigma_\Delta < 0$; hence also, thanks to properties of the Kronecker product,

$$\begin{aligned}
\Sigma_L &= \text{diag}(I \otimes T, I \otimes T, I)\text{diag}(\Sigma_\Delta, \Sigma_N) \\
&\qquad \times \text{diag}(I \otimes T^T, I_N \otimes T^T, I) < 0
\end{aligned}$$

If the functional V is defined as $V(\xi) = \xi^T (I \otimes Q^{-T} \bar{P} Q^{-1})\xi$, then according to the Lemma 4, $\dot{V}(\xi) \leq -\alpha \|\xi\|^2 + \gamma_5(1 + \varepsilon)(\eta^T (I \otimes E_3^T E_3)\eta + \omega^T \omega)$. Then, set $q_1 = \gamma_5(1 + \varepsilon)\|(I \otimes E_3^T E_3\|$ and $q_2 = \gamma_5(1 + \varepsilon)$. $\qquad \square$

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BRANISLAV REHÁK was born in Jilemnice, Czech Republic, in 1972. He received the M.S. degree in cybernetics from Czech Technical University, in 1995, the M.S. degree in mathematics from Charles University, Prague, in 1995, and the Ph.D. degree in control engineering and robotics from Czech Technical University, in 2004.

Since 2005, he has been a Researcher with The Czech Academy of Sciences, Institute of Information Theory and Automation, Prague, Czech Republic. His research interests include nonlinear control, control of time-delay systems, control of large-scale systems, and synchronization of multi-agent systems and complex networks. He is an Associate Editor of the *Kybernetika* journal.

VOLODYMYR LYNNYK was born in Lebedyn, Ukraine, in 1981. He received the B.S. and M.S. degrees in electronics from Sumy State University, Sumy, Ukraine, in 2004, and the Ph.D. degree in control engineering and robotics from Czech Technical University, Prague, Czech Republic, in 2010.

Since 2004, he has been a Researcher with the Control Theory Department, The Czech Academy of Sciences, Institute of Information Theory and Automation, Prague. He joined Czech Technical University, in 2018, where he is currently a Researcher. His research interests include the control of nonlinear systems, chaotic systems, biological systems, complex networks, and multi-agent systems.

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