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Leader-following synchronization of a multi-agent system with heterogeneous delays*

Branislav REHÁK‡, Volodymyr LYNNYK

The Czech Academy of Sciences, Institute of Information Theory and Automation, Praha 18200, Czech Republic

E-mail: rehakb@utia.cas.cz; volodymyr.lynnyk@utia.cas.cz

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Abstract: An algorithm is presented for leader-following synchronization of a multi-agent system composed of linear agents with time delay. The presence of different delays in various agents can cause a synchronization error that does not converge to zero. However, the norm of this error can be bounded and this boundary is presented. The proof of the main results is formulated by means of linear matrix inequalities, and the size of this problem is independent of the number of agents. Results are illustrated through examples, highlighting the fact that the steady error is caused by heterogeneous delays and demonstrating the capability of the proposed algorithm to achieve synchronization up to a certain error.

Key words: Multi-agent system; Time delay; Linear matrix inequality
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1 Introduction

The synchronization of multi-agent systems is a field of recent control theory with numerous practical applications. Two basic problems, the leader-following problem and the consensus problem, can be distinguished. To solve the leader-following problem, it is supposed that one special agent (the so-called leader) exists whose dynamics is not influenced by other agents, and that the other agents should replicate the behavior of the leader. In contrast, in the consensus problem, the agents should converge to a common behavior. For comparison, readers can refer to Li ZK et al. (2010) and Ni and Cheng (2010). The characteristic feature of the multi-agent synchronization problem is limited by communication between agents. Furthermore, the synchronization of dynamical networks was investigated by Anzo-Hernández et al. (2019); attention was paid to parameter mismatch, which is a problem closely related to the problem studied here.

Delays are inevitable in control over communication networks due to the transmission of data in data packets and the packet dropouts. Bakule et al. (2016) and Rehák and Lynnyk (2019a) dealt with large-scale system control over networks (a problem similar to the problem of synchronization of multi-agent systems). Hence, the control algorithm must be capable of dealing with these issues. To solve these problems, several algorithms have been proposed. Many of these algorithms use Lyapunov-Krasovskii or Razumikhin functions that allow the conversion of the stabilization problem of a time-delay system to the solution of a set of linear matrix inequalities (LMIs). For this purpose, the descriptor approach described in Fridman (2014) shows many benefits (easy implementation, not too conservative, and the ability to deal with fast varying delays). It is used in this study as well.

‡ Corresponding author

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ORCID: Branislav REHÁK, https://orcid.org/0000-0003-4264-4827; Volodymyr LYNNYK, https://orcid.org/0000-0002-6793-5324

To solve the problem of synchronization of multi-agent systems over communication networks, these challenges must be overcome. In many cases, attention is restricted to multi-agent systems with homogeneous delays (all agents have equal delays). While this is a rather restrictive assumption, it simplifies analysis to a large extent. For example, the basic formulation of this problem and its solution can be found in Hou et al. (2017). Wang D et al. (2018) examined optimization of a multi-agent system with communication delay. The synchronization of systems with nonlinear agents was solved in Qian et al. (2019) using Lyapunov-Krasovskii functions similar to those used in this study. Zhou et al. (2018) dealt with the consensus of stochastic agents subject to time delays. Systems with sampling (which induce input delay) were treated in Wen et al. (2013), and a similar problem was solved in Li XJ and Yang (2017) for discrete-time agents. The consensus problem of nonlinear multi-agent systems with input delay was the topic of Rehák and Lynnyk (2019b), where a boundary on the synchronization error was found.

The control of systems with heterogeneous time delays is more challenging. However, there are several different types of synchronization (Lynnyk et al., 2019a, 2019b, 2020). Here, we deal with identical synchronization. As shown in the case of the consensus problem in Rehák and Lynnyk (2019c) (for symmetric graph topologies) and in Rehák and Lynnyk (2020) (for general topologies), the heterogeneous delays may lead to a steady synchronization error that does not converge to zero. However, the norm of this error can be estimated. The larger the difference in the delays of various agents, the larger the induced error. This problem was also investigated in Lin et al. (2012), Zhang LJ and Orosz (2017), and Meng et al. (2018). A similar problem of containment control under the presence of heterogeneous time delays was solved in Xie et al. (2018). Wang HL (2014), Petrillo et al. (2017), and Zhang MR et al. (2017) investigated the synchronization of multi-agent systems with different delays in every communication channel.

In this study, we present an algorithm for synchronization of a multi-agent system composed of identical agents with heterogeneous delays. It is shown that a synchronization error can arise, which does not converge to zero; thus, the identical synchronization is not achieved. A boundary of the norm of this steady error is derived through LMIs. In fact, the effect of non-equal time delays in agents can be regarded as disturbance and handled by methods known from the H_∞ control. The descriptor approach is used to deal with time delays and deliver an easy-to-implement algorithm.

Notations used in this paper are summarized as follows:

(1) Symbol $\boldsymbol{I}_k$ denotes a k-dimensional identity matrix, while the zero matrix is denoted by $\mathbf{0}$; its dimension is always clear from the context.

(2) The symbol $\|\cdot\|$ stands for the quadratic (Euclidean) norm (even for matrices).

(3) The time argument t is omitted for functions of time if it does not cause confusion; $f(t)$ indicates the same as f. If the argument is different from t, it is written.

(4) The time delay is expressed by the subscript: $f(t-\tau)=f_\tau(t)=f_\tau$.

(5) The linear matrix inequality $\boldsymbol{P}>0$ indicates that matrix $\boldsymbol{P}$ is symmetric positive definite.

(6) The blocks below the diagonal are replaced by an asterisk for symmetric matrices: $\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{b}^{\mathrm{T}} & \boldsymbol{c} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ * & \boldsymbol{c} \end{pmatrix}$.

(7) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are matrices, then $\mathrm{diag}(\boldsymbol{a},\ \boldsymbol{b}) = \begin{pmatrix} \boldsymbol{a} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b} \end{pmatrix}$.

(8) The Kronecker product is denoted by "$\otimes$."

2 Graph theory

An important tool for the analysis of multi-agent systems is graph theory; therefore, the most important facts are presented. For detailed information, readers can refer to Li ZK et al. (2010). Let N be a positive integer. Then, define $\mathcal{V}=\{0,1,\ldots,N\}$ and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$. The graph $\mathcal{G}$ is defined as $\mathcal{G}=(\mathcal{V},\mathcal{E})$. Here, $\mathcal{V}$ is called the set of vertices and $\mathcal{E}$ the set of edges. If there is an edge from node j to node i, then $(i,j)\in\mathcal{E}$. When translated into the multi-agent system terminology, there is a connection from agent j to agent i: $E_{ji}=1$. Hence, the state of agent j is required to compute the control fed to agent i. It is assumed that for all $i=0,1,\cdots,N$, $E_{ii}=0$ holds.

Assume $k_0\in\{0,1,\ldots,N-1\}$ and assume the existence of a sequence $(i_k,i_{k+1})\in\mathcal{E}$ with $k=0,1,\ldots,k_0$. This sequence is called the directed path from node i_0 to node i_{k_0+1}. Graph $\mathcal{G}$ contains a spanning tree if node $i\in\mathcal{V}$ exists such that for every node $j\in\mathcal{V}$ $(j\neq i)$, a directed path from node i to node j can be found. If there is a node i_0 such that for any node $j \neq i_0$, there exists a directed path from i_0 to j but no path exists from any node to i_0, then we call the associated agent the leader or the root of the spanning tree. The leader is unique. Without loss of generality, we suppose that the leader agent is denoted by "0."

Given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we define the adjacency matrix $\boldsymbol{E} \in \mathbb{R}^{N\times N}$ as $E_{ij} = 1$ if $(i,j) \in \mathcal{E}$, and $E_{ij} = 0$ otherwise. We assume that graph $\mathcal{G}$ contains no loops, which indicates that for all $i = 1, 2, \ldots, N$, $E_{ii} = 0$ holds. Finally, we define the Laplacian matrix $\boldsymbol{L} \in \mathbb{R}^{N\times N}$ by $L_{ij} = -E_{ij}$ for $i, j = 1, 2, \ldots, N$ $(i \neq j)$, and $L_{ii} = -\sum_{j=1}^{N} E_{ij}$.

3 Problem setting

The multi-agent system considered in this study is composed of the following agents:

$$\begin{cases} \dot{\boldsymbol{x}}_0 = \boldsymbol{A}\boldsymbol{x}_0, \\ \dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\boldsymbol{u}_i, \ i = 1, 2, \ldots, N, \end{cases} \tag{1}$$

$$\boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}, \quad i = 0, 1, \ldots, N. \tag{2}$$

The matrices and vectors have the following dimensions: $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$.

For $i = 1, 2, \ldots, N$, define d_i as $d_i = 1$ if agent i receives information from the leader, and $d_i = 0$ otherwise. The control input $\boldsymbol{u}_i$ of agent i is expressed as (Ni and Cheng, 2010)

$$\boldsymbol{u}_i = -\boldsymbol{K} d_i(\boldsymbol{x}_{0,\tau_0} - \boldsymbol{x}_{i,\tau_i}) - \sum_{j=1}^{N} E_{ij}\boldsymbol{K}(\boldsymbol{x}_{j,\tau_j} - \boldsymbol{x}_{i,\tau_i}), \tag{3}$$

where matrix $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ is designed and the function $\tau_i : [0, \infty) \to [0, \infty)$ is the time delay associated with agent i. Note that these delays are not identically equal. The investigation of such systems is the purpose of this study.

Assumption 1 Time delays satisfy the following requirements:

(1) Functions τ_i are measurable.

(2) A constant $\bar{\tau} > 0$ exists such that for $i = 0, 1, \ldots, N$ and $t \geq 0$, $\tau_i(t) \leq \bar{\tau}$ holds.

The time delay τ_i represents the time needed to display information of agent i to its neighbor agents. Define $\bar{\boldsymbol{x}} = \dfrac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i$.

4 Error dynamics

For future examination, define vectors $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^{Nn}$ as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_N \end{pmatrix}, \ \boldsymbol{v} = \begin{pmatrix} \int_{t-\tau_1}^{t} \dot{\boldsymbol{x}}_1(s)\mathrm{d}s \\ \int_{t-\tau_2}^{t} \dot{\boldsymbol{x}}_2(s)\mathrm{d}s \\ \vdots \\ \int_{t-\tau_N}^{t} \dot{\boldsymbol{x}}_N(s)\mathrm{d}s \end{pmatrix}, \tag{4}$$

and analogously, $\boldsymbol{v}_0 = \int_{t-\tau_0}^{t} \dot{\boldsymbol{x}}_0(s)\mathrm{d}s$. With $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$, the overall multi-agent system can be written in compact form as

$$\begin{aligned} \dot{\boldsymbol{x}} =& (\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{x} - (\boldsymbol{L} \otimes \boldsymbol{BK})\boldsymbol{x} + (\boldsymbol{L} \otimes \boldsymbol{BK})\boldsymbol{v} \\ &+ (\boldsymbol{D} \otimes \boldsymbol{BK})[\boldsymbol{a} \otimes (\boldsymbol{x}_0 - \boldsymbol{v}_0) - \boldsymbol{x} + \boldsymbol{v}] \\ =& (\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{x} - [(\boldsymbol{L} + \boldsymbol{D}) \otimes \boldsymbol{BK}]\boldsymbol{x} \\ &+ [(\boldsymbol{L} + \boldsymbol{D}) \otimes \boldsymbol{BK}]\boldsymbol{v} \\ &+ (\boldsymbol{D} \otimes \boldsymbol{BK})[\boldsymbol{a} \otimes (\boldsymbol{x}_0 - \boldsymbol{v}_0)]. \end{aligned} \tag{5}$$

For example, one can find Proposition 1 (Li ZK et al., 2010; Xu et al., 2014):

Proposition 1 Matrix $\boldsymbol{L}+\boldsymbol{D}$ has eigenvalues with positive real parts.

The disagreement vector which is useful in the subsequent text is denoted by $\boldsymbol{\xi}$. Using $\boldsymbol{a} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^N$, it is defined as

$$\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{a} \otimes \boldsymbol{x}_0. \tag{6}$$

Furthermore, $\boldsymbol{\xi} = \boldsymbol{0}$ implies that $\boldsymbol{x}_0 = \boldsymbol{x}_1 = \ldots = \boldsymbol{x}_N$. Moreover, define the following two vectors:

$$\boldsymbol{\omega}_1 = \begin{pmatrix} \int_{t-\tau_0}^{t} \dot{\boldsymbol{\xi}}_1(s)\mathrm{d}s \\ \int_{t-\tau_0}^{t} \dot{\boldsymbol{\xi}}_2(s)\mathrm{d}s \\ \vdots \\ \int_{t-\tau_0}^{t} \dot{\boldsymbol{\xi}}_N(s)\mathrm{d}s \end{pmatrix}, \ \boldsymbol{\omega}_2 = \begin{pmatrix} \int_{t-\tau_1}^{t-\tau_0} \dot{\boldsymbol{x}}_1(s)\mathrm{d}s \\ \int_{t-\tau_2}^{t-\tau_0} \dot{\boldsymbol{x}}_2(s)\mathrm{d}s \\ \vdots \\ \int_{t-\tau_N}^{t-\tau_0} \dot{\boldsymbol{x}}_N(s)\mathrm{d}s \end{pmatrix}. \tag{7}$$

The goal is to design matrix $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ and find a constant $\varkappa > 0$ such that for any $T' > 0$,

$$\lim_{t\to\infty} \sum_{i=1}^{N} \int_0^{T'} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_0(t)\|^2 \mathrm{d}t \leq \varkappa \int_0^{T'} \|\boldsymbol{\omega}_2(t)\|^2 \mathrm{d}t \tag{8}$$

holds, if $\boldsymbol{x}_i(t) = \boldsymbol{x}_0(t)$ for $t \in [-\bar{\tau}, 0]$.

Remark 1 Inequality (8) is between H_∞-norms of the synchronization error and vector $\boldsymbol{\omega}_2$. However, this vector is zero if all delays are equal. Hence, it is regarded as a disturbance caused by dealing with non-equal delays in agents.

Lemma 1 Vector $\boldsymbol{\xi}$ obeys the dynamics given by

$$\dot{\boldsymbol{\xi}} = (\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{\xi} - [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}](\boldsymbol{\xi} - \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2).$$

Proof Note that $\boldsymbol{v} - \boldsymbol{a} \otimes \boldsymbol{v}_0 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ and $\boldsymbol{La} = \boldsymbol{0}$. The latter relationship implies $(\boldsymbol{L} \otimes \boldsymbol{BK})\boldsymbol{x} = (\boldsymbol{L} \otimes \boldsymbol{BK})\boldsymbol{\xi}$. These relationships will be used in the following reasoning:

$$\begin{aligned}\dot{\boldsymbol{\xi}} =& \dot{\boldsymbol{x}} - \boldsymbol{a} \otimes \dot{\boldsymbol{x}}_0\\
=&(\boldsymbol{I}_N \otimes \boldsymbol{A})(\boldsymbol{x} - \boldsymbol{a} \otimes \boldsymbol{x}_0) - [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}]\boldsymbol{x}\\
&+ [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}](\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{a} \otimes \boldsymbol{v}_0)\\
&+ (\boldsymbol{D} \otimes \boldsymbol{BK})[\boldsymbol{a} \otimes (\boldsymbol{x}_0 - \boldsymbol{v}_0)]\\
=&(\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{\xi} - (\boldsymbol{L} \otimes \boldsymbol{BK})\boldsymbol{\xi} - (\boldsymbol{D} \otimes \boldsymbol{BK})\boldsymbol{x}\\
&+ [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}](\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\\
&+ (\boldsymbol{D} \otimes \boldsymbol{BK})(\boldsymbol{a} \otimes \boldsymbol{v}_0)\\
&+ (\boldsymbol{D} \otimes \boldsymbol{BK})[\boldsymbol{a} \otimes (\boldsymbol{x}_0 - \boldsymbol{v}_0)]\\
=&(\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{\xi} - [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}]\boldsymbol{\xi}\\
&+ [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}](\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2).\end{aligned}$$

Remark 2 Note that the term $(\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{\xi} - [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}]\boldsymbol{\xi} + [(\boldsymbol{L}+\boldsymbol{D}) \otimes \boldsymbol{BK}]\boldsymbol{\omega}_1$ depends on the error $\boldsymbol{\xi}$. However, $\boldsymbol{\omega}_2$ depends directly only on the derivatives of state $\boldsymbol{x}_i$ for $i \neq 0$. Also, note that $\boldsymbol{\omega}_2 = \boldsymbol{0}$ if all delays in the multi-agent system are equal. This term is also a perturbation of the error dynamics due to the unequal time delays.

5 Leader-following synchronization

Another transformation is introduced at the beginning of this section. Assume that there exists a non-singular matrix $\boldsymbol{T} \in \mathbb{R}^{N\times N}$ so that $\boldsymbol{L} = \boldsymbol{T}^{-1}\boldsymbol{JT}$. Thus, matrix $\boldsymbol{J}$ has a Jordan structure.

Let $\boldsymbol{\zeta} = (\boldsymbol{T} \otimes \boldsymbol{I}_N)\boldsymbol{\xi}$, $\boldsymbol{\eta}_1 = (\boldsymbol{T} \otimes \boldsymbol{I}_N)\boldsymbol{\omega}_1$, and $\boldsymbol{\eta}_2 = (\boldsymbol{T} \otimes \boldsymbol{I}_N)\boldsymbol{\omega}_2$. We decompose vectors $\boldsymbol{\zeta}$, $\boldsymbol{\eta}_1$, and $\boldsymbol{\eta}_2$ into N-tuple of n-dimensional vectors: $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1^{\mathrm{T}}, \boldsymbol{\zeta}_2^{\mathrm{T}}, \ldots, \boldsymbol{\zeta}_N^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{\eta}_1 = (\boldsymbol{\eta}_{1,1}^{\mathrm{T}}, \boldsymbol{\eta}_{1,2}^{\mathrm{T}}, \ldots, \boldsymbol{\eta}_{1,N}^{\mathrm{T}})^{\mathrm{T}}$, and $\boldsymbol{\eta}_2 = (\boldsymbol{\eta}_{2,1}^{\mathrm{T}}, \boldsymbol{\eta}_{2,2}^{\mathrm{T}}, \ldots, \boldsymbol{\eta}_{2,N}^{\mathrm{T}})^{\mathrm{T}}$. Then, Eq. (6) is transformed into the $\boldsymbol{\zeta}$-variable form:

$$\dot{\boldsymbol{\zeta}} = (\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{\zeta} - (\boldsymbol{J} \otimes \boldsymbol{BK})(\boldsymbol{\zeta} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2). \quad (9)$$

The matrix $\boldsymbol{\Sigma}$ defined in the Appendix is needed here. With these tools, the following lemma can be proven:

Lemma 2 Let $\boldsymbol{J}$ be composed of real Jordan blocks with dimension one so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Let $\boldsymbol{Q}$, $\boldsymbol{S}$, and $\boldsymbol{W}$ be matrices of dimension $n \times n$ such that $\boldsymbol{S} > 0$, $\boldsymbol{W} > 0$, and $\boldsymbol{Q}$ is non-singular. Moreover, assume that $\boldsymbol{Y} \in \mathbb{R}^{m\times n}$, $\varepsilon > 0$, and $\gamma > 0$ exist, so that $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_1\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma) < 0$ and $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_N\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma) < 0$ hold. Then with $\boldsymbol{K} = \boldsymbol{YQ}^{-1}$, for any $T' > 0$, if $\boldsymbol{\zeta}(t) = \boldsymbol{0}$ for $t \in [-\bar{\tau}, 0]$, $\int_0^{T'} \|\boldsymbol{\zeta}\|^2 \mathrm{d}t \leq \gamma(1+\varepsilon)\lambda_N^2 \|\boldsymbol{BK}\|^2 \int_0^{T'} \|\boldsymbol{\omega}_2\|^2 \mathrm{d}t$ holds.

Proof System (9) consists of N equations:

$$\dot{\boldsymbol{\zeta}}_i = \boldsymbol{A\zeta}_i - \lambda_i \boldsymbol{BK}(\boldsymbol{\zeta}_i - \boldsymbol{\eta}_{1,i} - \boldsymbol{\eta}_{2,i}). \quad (10)$$

Since $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma)$ is convex in λ, if $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_1\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma) < 0$ and $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_N\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma) < 0$ are valid, then $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_i\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma) < 0$ holds for all λ_i ($i = 1, 2, \ldots, N$).

Using Remark A1 in the Appendix, considering the ordering of the eigenvalues λ_i, one can obtain (for any $T' > 0$ and zero initial conditions)

$$\int_0^{T'} \|\boldsymbol{\zeta}_i\|^2 \mathrm{d}t \leq \gamma(1+\varepsilon)\lambda_N^2 \|\boldsymbol{BK}\|^2 \int_0^{T'} \|\boldsymbol{\eta}_{2,i}\|^2 \mathrm{d}t, \quad (11)$$

where $i = 1, 2, \ldots, N$.

Lemma 3 deals with the case of $\boldsymbol{L}+\boldsymbol{D}$ with one Jordan block with real eigenvalues:

Lemma 3 Assume that

$$\boldsymbol{J} = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0\\ 0 & \lambda & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \ldots & 1\\ 0 & 0 & 0 & \ldots & \lambda \end{pmatrix}$$

such that $\boldsymbol{J} \in \mathbb{R}^{N\times N}$. Let $\boldsymbol{Q}$, $\boldsymbol{S}$, and $\boldsymbol{W}$ be matrices with dimension $n \times n$ such that $\boldsymbol{S} > 0$, $\boldsymbol{W} > 0$, and $\boldsymbol{Q}$ is non-singular. Moreover, assume that $\boldsymbol{Y} \in \mathbb{R}^{m\times n}$, $\varepsilon > 0$, and $\gamma > 0$ exist, such that $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_1\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}, \varepsilon, \gamma) < 0$ and $\boldsymbol{K} = \boldsymbol{YQ}^{-1}$ hold. Then, a constant $C_\lambda > 0$ exists such that

$$\int_0^{T'} \|\boldsymbol{\zeta}\|^2 \mathrm{d}t \leq C_\lambda \int_0^{T'} \|\boldsymbol{\eta}_2\|^2 \mathrm{d}t, \quad (12)$$

if $\boldsymbol{\zeta}(t) = \boldsymbol{0}$ for $t \in [-\bar{\tau}, 0]$.

Proof Lemma 2 implies that $\int_0^{T'} \|\boldsymbol{\zeta}_N\|^2 \mathrm{d}t \leq \gamma(1+\varepsilon)\lambda^2 \|\boldsymbol{BK}\|^2 \int_0^{T'} \|\boldsymbol{\eta}_{2,N}\|^2 \mathrm{d}t$. Assume that inequality (12) is valid for $j = N, N-1, \ldots, i+1$. The structure of matrix $\boldsymbol{J}$ implies

$$\dot{\boldsymbol{\zeta}}_i = \boldsymbol{A\zeta}_i - \lambda\boldsymbol{BK}(\boldsymbol{\zeta}_i - \boldsymbol{\eta}_{1,i} - \boldsymbol{\eta}_{2,i}) + \lambda\boldsymbol{BK\zeta}_{i+1}. \quad (13)$$

Applying Lemma A1 on Eq. (10) with $\boldsymbol{w}_i = -\lambda\boldsymbol{BK}\boldsymbol{\zeta}_{i+1} + \boldsymbol{\eta}_{2,i}$, we then obtain

$$\int_0^{T'} \|\boldsymbol{\zeta}_i\|^2\mathrm{d}t \leq \gamma(1+\varepsilon)\lambda^2\|\boldsymbol{BK}\|^2\|\boldsymbol{\eta}_{2,i}-\lambda\boldsymbol{BK}\boldsymbol{\zeta}_{i+1}\|^2.$$

Hence, a function $C : \mathbb{N} \times [0,\infty) \times [0,\infty) \to [0,\infty)$ exists such that for positive constants λ', γ', and ε' and any $i \in \mathbb{N}$, $C(i,\lambda',\gamma',\varepsilon') \leq C(i-1,\lambda',\gamma',\varepsilon')$ and

$$\int_0^{T'} \|\boldsymbol{\zeta}_i\|^2\mathrm{d}t \leq C(i,\lambda\|\boldsymbol{BK}\|,\gamma,\varepsilon)\left\|\begin{pmatrix}\boldsymbol{\eta}_{2,i}\\ \boldsymbol{\eta}_{2,2}\\ \vdots\\ \boldsymbol{\eta}_{2,N}\end{pmatrix}\right\|^2$$

hold.

Then, take $C_\lambda = C(1,\lambda\|\boldsymbol{BK}\|,\gamma,\varepsilon)$.

Now, consider the case of matrix $\boldsymbol{J}$ having Jordan blocks with imaginary eigenvalues. Here, we use a procedure similar to that in Zuo et al. (2017). Let matrix $(\boldsymbol{L}+\boldsymbol{D}) \in \mathbb{R}^{2\times 2}$ have two complex conjugated eigenvalues $\iota = \alpha + \mathrm{j}\beta$ and $\bar{\iota} = \alpha - \mathrm{j}\beta$. In this case, we can consider the Jordan block defined as $\boldsymbol{J}'(\iota) = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$.

Lemma 4 Suppose that matrix $(\boldsymbol{L}+\boldsymbol{D}) \in \mathbb{R}^{2\times 2}$ has two complex conjugated eigenvalues $\iota = \alpha + \mathrm{j}\beta$ and $\bar{\iota} = \alpha - \mathrm{j}\beta$. Assume that matrices $\boldsymbol{Q}$, $\boldsymbol{S}$, and $\boldsymbol{W}$ exist with dimension $n \times n$ such that $\boldsymbol{S} > 0$, $\boldsymbol{W} > 0$, and $\boldsymbol{Q}$ is non-singular. Moreover, assume that $\boldsymbol{Y} \in \mathbb{R}^{m\times n}$, $\varepsilon > 0$, and $\gamma > 0$ exist, such that $\boldsymbol{K} = \boldsymbol{YQ}^{-1}$ and $\boldsymbol{\varSigma}(\boldsymbol{I}_2 \otimes \boldsymbol{A}, \boldsymbol{J}'(\iota) \otimes \boldsymbol{B}, \boldsymbol{I}_2 \otimes \boldsymbol{Y}, \boldsymbol{I}_2 \otimes \boldsymbol{Q}, \boldsymbol{I}_2 \otimes \boldsymbol{S}, \boldsymbol{I}_2 \otimes \boldsymbol{W}, \varepsilon, \gamma) < 0$ hold. Then, for any $T' > 0$,

$$\int_0^{T'} \|(\boldsymbol{\zeta}_1^{\mathrm{T}}, \boldsymbol{\zeta}_2^{\mathrm{T}})\|^2\mathrm{d}t$$
$$\leq \gamma(1+\varepsilon)\|\boldsymbol{J}'(\iota)\|^2\|\boldsymbol{BK}\|^2\int_0^{T'} \|\boldsymbol{\eta}_2\|^2\mathrm{d}t \qquad (14)$$

holds.

The proof is conducted analogously in the case of real eigenvalues. From Lemma A1, it follows that

$$\int_0^{T'} \|(\boldsymbol{\zeta}_1^{\mathrm{T}}, \boldsymbol{\zeta}_2^{\mathrm{T}})\|^2\mathrm{d}t$$
$$\leq \gamma(1+\varepsilon)\|\boldsymbol{J}'(\iota)\otimes\boldsymbol{BK}\|^2\int_0^{T'} \|\boldsymbol{\eta}_2\|^2\mathrm{d}t.$$

A useful property of the Kronecker product is derived in Lancaster and Farahat (1972): $\|\boldsymbol{J}'(\iota) \otimes \boldsymbol{BK}\| = \|\boldsymbol{J}'(\iota)\|\|\boldsymbol{BK}\|^2$. For a simple eigenvalue pair ι and $\bar{\iota}$, one can derive

$$\int_0^{T'} \|(\boldsymbol{\zeta}_1^{\mathrm{T}}, \boldsymbol{\zeta}_2^{\mathrm{T}})\|^2\mathrm{d}t$$
$$\leq \gamma(1+\varepsilon)\|\boldsymbol{J}'(\iota)\|^2\|\boldsymbol{BK}\|^2\int_0^{T'} \|\boldsymbol{\eta}_2\|^2\mathrm{d}t. \qquad (15)$$

The case of several simple eigenvalues is straightforward, whereas in the presence of complex eigenvalues in the Jordan structure, an iterative procedure analogous to the one presented in the case of multiple real eigenvalues can be derived.

Lemma 5 Assume that N is an even integer. Let matrix $(\boldsymbol{L}+\boldsymbol{D}) \in \mathbb{R}^{2\times 2}$ be composed of a block corresponding to two complex conjugated eigenvalues $\iota = \alpha + \mathrm{j}\beta$ and $\bar{\iota} = \alpha - \mathrm{j}\beta$ with multiplicity $N' = N/2$. Moreover, assume that $n \times n$-dimensional matrices $\boldsymbol{Q}$, $\boldsymbol{S}$, and $\boldsymbol{W}$ exist with $\boldsymbol{S} > 0$, $\boldsymbol{W} > 0$, and $\boldsymbol{Q}$ being non-singular, and that $\boldsymbol{Y} \in \mathbb{R}^{m\times n}$, $\varepsilon > 0$, and $\gamma > 0$ exist, such that $\boldsymbol{K} = \boldsymbol{YQ}^{-1}$ and $\boldsymbol{\varSigma}(\boldsymbol{I}_2 \otimes \boldsymbol{A}, \boldsymbol{J}'(\iota) \otimes \boldsymbol{B}, \boldsymbol{I}_2 \otimes Y, \boldsymbol{I}_2 \otimes \boldsymbol{Q}, \boldsymbol{I}_2 \otimes \boldsymbol{S}, \boldsymbol{I}_2 \otimes \boldsymbol{W}, \varepsilon, \gamma) < 0$ hold. Then, a constant $C_{\alpha,\beta}$ exists so that for any $T' > 0$,

$$\int_0^{T'} \|(\boldsymbol{\zeta}_i^{\mathrm{T}}, \boldsymbol{\zeta}_{i+1}^{\mathrm{T}})\|^2\mathrm{d}t \leq C_{\alpha,\beta}\|\boldsymbol{\eta}_2\|^2\mathrm{d}t \qquad (16)$$

holds.

Proof First, note that

$$\int_0^{T'} \|(\boldsymbol{\zeta}_{N-1}^{\mathrm{T}}, \boldsymbol{\zeta}_N^{\mathrm{T}})\|^2\mathrm{d}t$$
$$\leq \gamma(1+\varepsilon)\|\boldsymbol{J}'(\iota)\|^2\|\boldsymbol{BK}\|^2\int_0^{T'} \left\|\begin{pmatrix}\boldsymbol{\eta}_{2,N-1}\\ \boldsymbol{\eta}_{2,N}\end{pmatrix}\right\|^2\mathrm{d}t. \qquad (17)$$

Proceeding similar to the case of multiple real eigenvalues, one infers that there exists a function $C' : \mathbb{N} \times [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for any positive constants λ', γ', and ε' and any $i \in \mathbb{N}$, $C'(i,\lambda',\gamma',\varepsilon') \leq C'(i-1,\lambda',\gamma',\varepsilon')$ and

$$\int_0^{T'} \|(\boldsymbol{\zeta}_{2i}, \boldsymbol{\zeta}_{2i+1})\|^2\mathrm{d}t$$
$$\leq C\big(i,\|\boldsymbol{J}'(\iota)\|\|\boldsymbol{BK}\|,\gamma,\varepsilon\big)\big\|[\boldsymbol{\eta}_{2,2i}^{\mathrm{T}}, \boldsymbol{\eta}_{2,2i+1}^{\mathrm{T}}, \ldots, \boldsymbol{\eta}_{2,N}^{\mathrm{T}}]^{\mathrm{T}}\big\|^2$$

hold.

Then, take $C_{\alpha,\beta} = C(1, \|\boldsymbol{J}'(\iota)\|\|\boldsymbol{BK}\|, \gamma, \varepsilon)$. In summary, one arrives at the following theorem:

Theorem 1 Let constants $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\iota_1, \iota_2, \ldots, \iota_k \in \mathbb{C}$ such that all eigenvalues of $\boldsymbol{L}+\boldsymbol{D}$

lie in the convex hull of $\lambda_1, \lambda_2, \iota_1, \bar{\iota}_1, \ldots, \iota_k, \bar{\iota}_k$. Assume that $n \times n$-dimensional matrices $\boldsymbol{Q}$, $\boldsymbol{S}$, and $\boldsymbol{W}$ exist, such that $\boldsymbol{S} > 0$, $\boldsymbol{W} > 0$, and $\boldsymbol{Q}$ is non-singular. Moreover, assume that $\boldsymbol{Y} \in \mathbb{R}^{m\times n}$, $\varepsilon > 0$, $\gamma > 0$ exist, so that with $\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{Q}^{-1}$, $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_i\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}) < 0$ holds for $i = 1, 2$ and $\boldsymbol{\Sigma}(\boldsymbol{I}_2 \otimes \boldsymbol{A}, \boldsymbol{J}'(\iota) \otimes \boldsymbol{B}, \boldsymbol{I}_2 \otimes \boldsymbol{Y}, \boldsymbol{I}_2 \otimes \boldsymbol{Q}, \boldsymbol{I}_2 \otimes \boldsymbol{S}, \boldsymbol{I}_2 \otimes \boldsymbol{W}, \varepsilon, \gamma) < 0$ holds for $i = 1, 2, \ldots, k$. Then, a constant $\varkappa$ exists such that for all $T' > 0$,

$$\int_0^{T'} \|\boldsymbol{\xi}\|\mathrm{d}t \le \varkappa \int_0^{T'} \|\boldsymbol{\omega}_2\|^2\mathrm{d}t \tag{18}$$

holds, if $\boldsymbol{\xi}(t) = \mathbf{0}$ for $t \in [-\bar{\tau}, 0]$.

Proof Due to the assumption in Theorem 1, constant $\alpha_i \in [0, 1]$ $(i = 1, 2, \ldots, k + 2)$ exist so that $\iota = \sum_{i=1}^{k} \alpha_i\iota_i + \alpha_{k+1}\lambda_1 + \alpha_{k+2}\lambda_2$. Also, observe that $\boldsymbol{\Sigma}(\boldsymbol{A}, \lambda_i\boldsymbol{B}, \boldsymbol{Y}, \boldsymbol{Q}, \boldsymbol{S}, \boldsymbol{W}) < 0$ for $i = 1, 2$ and $\boldsymbol{\Sigma}(\boldsymbol{I}_2\otimes\boldsymbol{A}, \lambda_i\boldsymbol{I}_2\otimes\boldsymbol{B}, \boldsymbol{I}_2\otimes\boldsymbol{Y}, \boldsymbol{I}_2\otimes\boldsymbol{Q}, \boldsymbol{I}_2\otimes\boldsymbol{S}, \boldsymbol{I}_2\otimes\boldsymbol{W}) < 0$ for $i = 1, 2$. Moreover, if λ_1 and λ_2 are regarded as complex numbers with a zero imaginary part, then $\boldsymbol{J}(\lambda_i) = \lambda_i\boldsymbol{I}_2$.

$$\boldsymbol{J}'(\iota) = \sum_{i=1}^{k} \alpha_i\boldsymbol{J}'(\iota_i) + \alpha_{k+1}\boldsymbol{J}'(\lambda_1) + \alpha_{k+2}\boldsymbol{J}'(\lambda_2). \tag{19}$$

Due to the assumption in Theorem 1, $\boldsymbol{J}'(\iota_i) < 0$ for $i = 1, 2, \ldots, k$, $\boldsymbol{J}'(\lambda_i) < 0$ for $i = 1, 2$, and $\boldsymbol{J}'(\iota) < 0$.

If λ_1 is a simple eigenvalue, then define $C_{\lambda_1} = \gamma(1 + \varepsilon)\lambda_1^2\|\boldsymbol{BK}\|^2$. Similarly, if λ_2 is a simple eigenvalue, then define $C_{\lambda_2} = \gamma(1 + \varepsilon)\lambda_2^2\|\boldsymbol{BK}\|^2$. If $k' \in \{1, 2, \ldots, k\}$, the eigenvalue $\iota_{k'} = \alpha_{k'} + \mathrm{j}\beta_{k'}$ is simple. Then define $C_{\alpha_{k'},\beta_{k'}} = \gamma(1 + \varepsilon)\|\boldsymbol{J}'(\iota_{k'})\|^2\|\boldsymbol{BK}\|^2$. Finally, let

$$\varkappa' = \max(C_{\lambda_1}, C_{\lambda_2}, C_{\alpha_1,\beta_1}, \ldots, C_{\alpha_k,\beta_k}).$$

Then, for $T' > 0$,

$$\int_0^{T'} \|\boldsymbol{\zeta}\|^2\mathrm{d}t \le \varkappa' \int_0^{T'} \|\boldsymbol{\eta}_2\|^2\mathrm{d}t \tag{20}$$

holds.

The non-singularity of matrix $\boldsymbol{T}$ implies the existence of a constant $\varkappa$ so that inequality (18) holds.

6 Examples

6.1 Example 1

In this example, two agents are described by

$$\dot{x}_{i,1} = x_{i,2},\ \dot{x}_{i,2} = x_{i,1} + b_iu_i,\ i = 0, 1,$$

where $b_0 = 0$ and $b_1 = 1$. The control input (3) is given as $u = (0.1760, 0.1767)(\boldsymbol{x}_0 - \boldsymbol{x}_{1,\tau_1})$. The delay is chosen as $\tau_1 = \pi/12$.

The norm of the synchronization error on a long time period is shown in Fig. 1. The error does not converge to zero due to presence of τ_1. However, if no delays are present, the error converges to zero.

As shown in this example, identical synchronization is not achieved; however, the amplitudes of the sine curves generated by the leader and the follower are equal, and the phase difference between these signals is constant, determined by the delays.

6.2 Example 2

Here, a network of six agents is studied, and the agents are again harmonic oscillators:

$$\dot{\boldsymbol{x}}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\boldsymbol{x}_i + \begin{pmatrix} 0 \\ b_i \end{pmatrix}\boldsymbol{u}_i,\ i = 0, 1, \ldots, 5,$$

where $b_0 = 0$ and $b_i = 1$ $(i = 1, 2, \ldots, 5)$. The interconnection of the agents is depicted in Fig. 2. The corresponding matrix $\boldsymbol{L} + \boldsymbol{D}$ has real eigenvalues in the interval $(0.13, 4.11)$. The maximal time delay $\bar{\tau} = 0.1$ s. By solving the matrix inequalities in Theorem 1, we obtain $\boldsymbol{K} = (0.2478,\ 0.2534)$. Delays $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ in the simulations are constant, and are set as 0.100, 0.015, 0.100, 0.050, 0.080, and 0.040 s, respectively. The initial conditions for agents are defined as $\boldsymbol{x}_0(t) = (1, 1)$, $\boldsymbol{x}_1(t) = (0, 0)$,

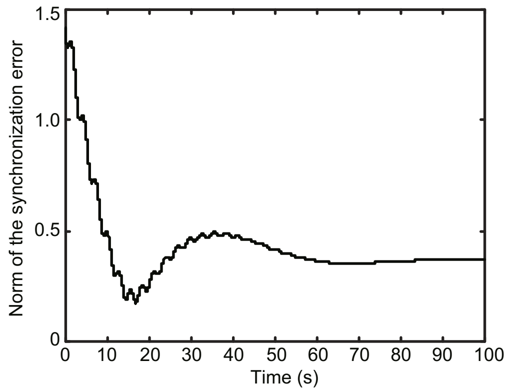

Fig. 1 Norm of the synchronization error in the presence of delays

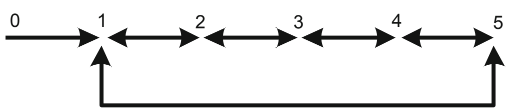

Fig. 2 Connection of agents

$\boldsymbol{x}_2(t) = (0, 0.5)$, $\boldsymbol{x}_3(t) = (0, 0)$, $\boldsymbol{x}_4(t) = (-1, 1)$, and $\boldsymbol{x}_5(t) = (-0.5, -0.5)$ for $t \in [-0.1, 0]$.

Results are shown in Figs. 3–5. In Fig. 3, the states $x_{0,1}$ for the leader (dotted line), $x_{1,1}$ for agent 1 (solid line), and $x_{3,1}$ for agent 3 (dashed line) are depicted. Moreover, the control inputs of agents 1 and 3 are depicted in Fig. 4, and the solid line and dashed line illustrate the control inputs of agents 1 and 3, respectively. The norm of the synchronization error over a long time period is shown in Fig. 5. Here, a steady error again appears. However, changing the time delay to 0.1 s in all agents results in a zero steady error, as illustrated in Fig. 6.

7 Conclusions and future work

An algorithm for leader-following synchronization of a multi-agent system has been proposed. The agents exhibiting time delays are, in general, different for different agents. It is shown that difference in delays causes a synchronization error that does not converge to zero. The norm of this error can be estimated, and the estimate is derived using linear matrix inequality optimization.

In the future, a more detailed study of the effect of heterogeneous time delays on identical synchronization is expected. The same algorithm will be used to obtain the solution to the more general problem, where delays differ in every communication link. Also, systems with nonlinearities or switching topology will be studied.

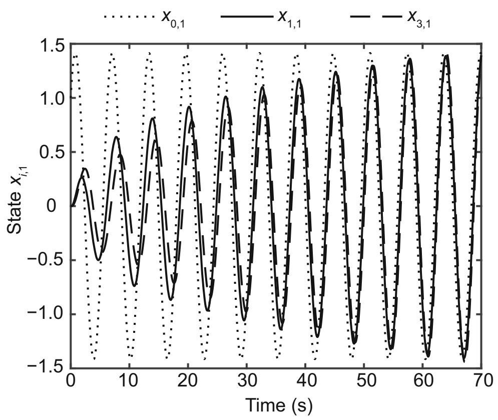

Fig. 3 State $\boldsymbol{x}_{i,1}$

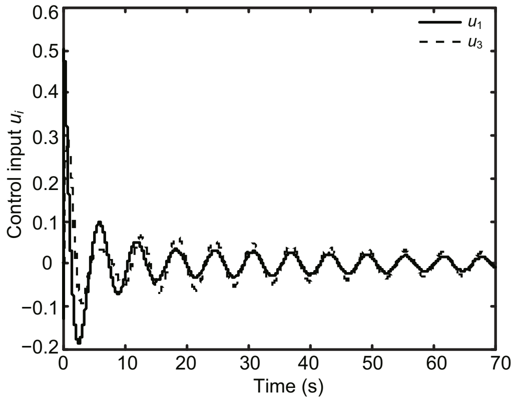

Fig. 4 Control input $\boldsymbol{u}_i$

Contributors

Branislav REHÁK designed the research. Branislav REHÁK and Volodymyr LYNNYK processed the data. Branislav REHÁK drafted the manuscript. Volodymyr LYNNYK helped organize the manuscript. Branislav REHÁK and Volodymyr LYNNYK revised and finalized the paper.

Compliance with ethics guidelines

Branislav REHÁK and Volodymyr LYNNYK declare that they have no conflict of interest.

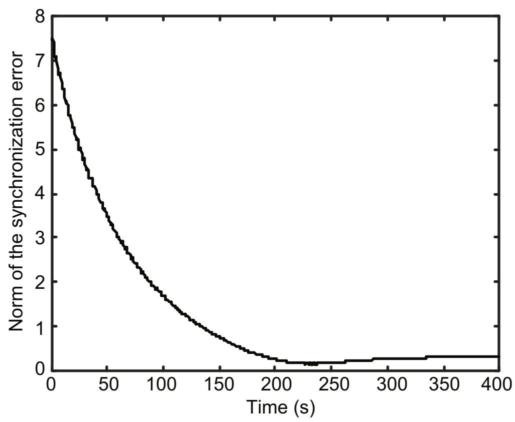

Fig. 5 Norm of the synchronization error with heterogeneous delays

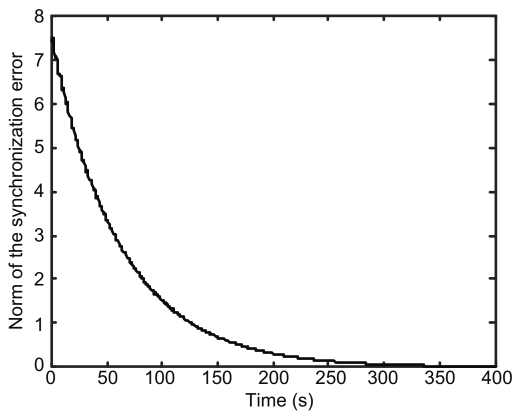

Fig. 6 Norm of the synchronization error with equal delays

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Appendix: Auxiliary results

In this section, an auxiliary problem is introduced and its solution is presented. A result concerning H_∞ stability of a time-delay system is presented, obtained through the descriptor approach (readers can refer to Fridman (2014) for details). For completeness, it is presented here together with its proof.

Let μ and ν be positive integers and matrices $\boldsymbol{A} \in \mathbb{R}^{\nu\times\nu}, \boldsymbol{B} \in \mathbb{R}^{\nu\times\mu}$, and $\boldsymbol{K} \in \mathbb{R}^{\mu\times\nu}$. We now consider the following auxiliary problem:

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{z}_\tau + \boldsymbol{B}\boldsymbol{K}\boldsymbol{w},\ \boldsymbol{z}(0) = \boldsymbol{z}_0, \tag{A1}$$

where $\boldsymbol{z}(0) \in \mathbb{R}^\nu$, the disturbance $\boldsymbol{w} : [0,\infty) \to \mathbb{R}^\nu$ is bounded, and the time delay has properties described above.

The following lemma holds for Eq. (A1):

Lemma A1 Assume that $\nu \times \nu$-dimensional matrices $\boldsymbol{Q}$, $\boldsymbol{S}$, and $\boldsymbol{W}$ exist, such that $\boldsymbol{S} > 0$, $\boldsymbol{W} > 0$, and $\boldsymbol{Q}$ is non-singular. Moveover, assume that $\boldsymbol{Y} \in \mathbb{R}^{\mu\times\nu}$, $\varepsilon > 0$, and $\gamma > 0$ exist, such that matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{6\nu\times 6\nu}$ is given by

$$\begin{aligned}&\boldsymbol{\Sigma}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{Y},\boldsymbol{Q},\boldsymbol{S},\boldsymbol{W},\varepsilon,\gamma)\\ &= \begin{pmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{12} & \boldsymbol{\sigma}_{13} & \boldsymbol{I}_\nu & \boldsymbol{0} & \boldsymbol{\sigma}_{16} \\ * & \boldsymbol{\sigma}_{22} & \varepsilon\boldsymbol{\sigma}_{13} & \boldsymbol{0} & \boldsymbol{I}_\nu & \boldsymbol{0} \\ * & * & \boldsymbol{\sigma}_{33} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\gamma\boldsymbol{I}_\nu & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & -\frac{\gamma}{\varepsilon}\boldsymbol{I}_\nu & \boldsymbol{0} \\ * & * & * & * & * & -\boldsymbol{I}_\nu \end{pmatrix},\end{aligned}$$

with $\boldsymbol{\sigma}_{11} = \boldsymbol{A}\boldsymbol{Q} + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} - (\boldsymbol{B}\boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}})$, $\boldsymbol{\sigma}_{12} = \boldsymbol{W} - \boldsymbol{Q} + \varepsilon\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$, $\boldsymbol{\sigma}_{13} = -\bar{\tau}\boldsymbol{B}\boldsymbol{Y}$, $\boldsymbol{\sigma}_{16} = \boldsymbol{Q}^{\mathrm{T}}$, $\boldsymbol{\sigma}_{22} = -\varepsilon(\boldsymbol{Q} + \boldsymbol{Q}^{\mathrm{T}} - \bar{\tau}\boldsymbol{S})$, and $\boldsymbol{\sigma}_{33} = -\bar{\tau}\boldsymbol{S}$. Thus,

$$\boldsymbol{\Sigma}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{Y},\boldsymbol{Q},\boldsymbol{S},\boldsymbol{W},\varepsilon,\gamma) < 0 \tag{A2}$$

holds.

Then, Eq. (A1) is asymptotically stable if $\boldsymbol{w} = \boldsymbol{0}$ for $t \geq 0$ with $\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{Q}^{-1}$. Otherwise, the following estimation holds:

$$\int_0^{T'} \|\boldsymbol{z}\|^2 \mathrm{d}t \leq \gamma(1+\varepsilon)\|\boldsymbol{B}\boldsymbol{K}\|^2 \int_0^{T'} \|\boldsymbol{w}\|^2 \mathrm{d}t, \tag{A3}$$

if $\boldsymbol{z}(t) = \boldsymbol{0}$ for $t \in [-\bar{\tau}, 0]$.

Proof The proof is divided into three steps. First, using a series of Schur complements, a matrix inequality condition equivalent to inequality (A2) is derived with a smaller dimension. Then, a row and column transformation is conducted and new matrix variables (matrices $\boldsymbol{V}$, $\boldsymbol{P}$, $\boldsymbol{R}$, and $\boldsymbol{K}$ in the subsequent text) are introduced, permitting construction of a Lyapunov-Krasovskii function used to infer H_∞ stability of the original system.

Step 1: define matrix $\boldsymbol{\Sigma}'$ (for simplification, the arguments are omitted) as

$$\boldsymbol{\Sigma}' = \begin{pmatrix} \boldsymbol{\sigma}'_{11} & \boldsymbol{\sigma}_{12} & \boldsymbol{\sigma}_{13} \\ * & \boldsymbol{\sigma}'_{22} & \varepsilon\boldsymbol{\sigma}_{13} \\ * & * & \boldsymbol{\sigma}_{33} \end{pmatrix},$$

with $\boldsymbol{\sigma}'_{11} = \boldsymbol{\sigma}_{11} + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} + \dfrac{1}{\gamma}\boldsymbol{I}_N$ and $\boldsymbol{\sigma}'_{22} = \boldsymbol{\sigma}_{22} + \dfrac{\varepsilon}{\gamma}\boldsymbol{I}_N$. Using the Schur complement on matrix $\boldsymbol{\Sigma}(\boldsymbol{A},\boldsymbol{B})$ three times, inequality (A2) is equivalent to

$$\boldsymbol{\Sigma}' < 0. \tag{A4}$$

Step 2: define $\boldsymbol{V} = \boldsymbol{Q}^{-1}$ (this is possible thanks to the assumption of non-singularity of $\boldsymbol{Q}$). Let

$$\begin{aligned}\boldsymbol{\Sigma}'' &= \mathrm{diag}(\boldsymbol{V}^{\mathrm{T}},\boldsymbol{V}^{\mathrm{T}},\boldsymbol{V}^{\mathrm{T}})\boldsymbol{\Sigma}'(\boldsymbol{A},\boldsymbol{B})\mathrm{diag}(\boldsymbol{V},\boldsymbol{V},\boldsymbol{V}),\\ \boldsymbol{P} &= \boldsymbol{V}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{V},\ \boldsymbol{R} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{V},\ \boldsymbol{K} = \boldsymbol{Y}\boldsymbol{V}.\end{aligned}$$

Step 3: for system (A1), we introduce the Lyapunov-Krasovskii function $\mathcal{W}$ by

$$\mathcal{W} = \boldsymbol{z}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{z} + \int_{-\bar{\tau}}^0 \int_{t+s}^t \dot{\boldsymbol{z}}^{\mathrm{T}}(\sigma)\boldsymbol{R}\dot{\boldsymbol{z}}(\sigma)\mathrm{d}\sigma. \tag{A5}$$

Define $\boldsymbol{\eta} = \dfrac{1}{\bar{\tau}}\int_{t-\bar{\tau}}^t \dot{\boldsymbol{z}}(s)\mathrm{d}s$ and note that $\boldsymbol{z}_\tau = \boldsymbol{z} - \bar{\tau}\boldsymbol{\eta}$. This implies $\dot{\mathcal{W}} \leq 2\dot{\boldsymbol{z}}\boldsymbol{P}\boldsymbol{z} - \bar{\tau}\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\eta} + \bar{\tau}\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{R}\dot{\boldsymbol{z}}$. Thus,

$$\begin{aligned}&\dot{\mathcal{W}} - \bar{\tau}\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\eta} + \bar{\tau}\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{R}\dot{\boldsymbol{z}} + \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z} - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}\\ \leq& 2\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{z} + (\boldsymbol{z}^{\mathrm{T}}\boldsymbol{V} + \varepsilon\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{V}^{\mathrm{T}})\\ &\cdot(-\dot{\boldsymbol{z}} + \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{z} - \bar{\tau}\boldsymbol{B}\boldsymbol{K}\boldsymbol{\eta} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{w})\\ &+ \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z} - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}\\ \leq& 2\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{z} + (\boldsymbol{z}^{\mathrm{T}}\boldsymbol{V}^{\mathrm{T}} + \varepsilon\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{V}^{\mathrm{T}})\\ &\cdot(-\dot{\boldsymbol{z}} + \boldsymbol{A}\boldsymbol{z} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{z} - \bar{\tau}\boldsymbol{B}\boldsymbol{K}\boldsymbol{\eta})\\ &+ \frac{1}{\gamma}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{z} + \frac{\varepsilon}{\gamma}\dot{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V}\dot{\boldsymbol{z}} + \boldsymbol{w}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{K}\boldsymbol{w}\\ \leq& (\boldsymbol{z}^{\mathrm{T}},\dot{\boldsymbol{z}}^{\mathrm{T}},\boldsymbol{\eta}^{\mathrm{T}})\boldsymbol{\Sigma}''(\boldsymbol{A},\boldsymbol{B})\begin{pmatrix}\boldsymbol{z}\\ \dot{\boldsymbol{z}}\\ \boldsymbol{\eta}\end{pmatrix}\\ &+ \gamma(1+\varepsilon)\|\boldsymbol{B}\boldsymbol{K}\|^2\|\boldsymbol{w}\|^2 - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}\\ <& \gamma(1+\varepsilon)\|\boldsymbol{B}\boldsymbol{K}\|^2\|\boldsymbol{w}\|^2 - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}.\end{aligned}$$

For any $T' > 0$, this yields

$$\int_0^{T'} \boldsymbol{z}^{\mathrm{T}}(s)\boldsymbol{z}(s)\mathrm{d}s \le \gamma(1+\varepsilon)\|\boldsymbol{BK}\|^2 \int_0^{T'} \boldsymbol{w}^{\mathrm{T}}(s)\boldsymbol{w}(s)\mathrm{d}s. \tag{A6}$$

Also, if $\boldsymbol{w} = \mathbf{0}$ for $t \ge 0$, we have $\dot{\mathcal{W}} < 0$, which implies the asymptotic stability. Otherwise, we obtain $\|\boldsymbol{z}\|^2 \le \gamma(1+\varepsilon)\|\boldsymbol{BK}\|^2\|\boldsymbol{w}\|_2^2$ for zero initial conditions.

Remark A1 Note that Eq. (A1) can be rewritten as

$$\dot{\boldsymbol{z}} = \boldsymbol{Az}+\boldsymbol{BKz}-\boldsymbol{BK}\int_{t-\tau}^{t} \dot{\boldsymbol{z}}(s)\mathrm{d}s+\boldsymbol{BKw},\ \boldsymbol{z}(0) = \boldsymbol{z}_0. \tag{A7}$$

This formulation is analogous to the form of Eq. (9).