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Generalized Lorenz Canonical Form Revisited*

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This paper completes the description of the generalized Lorenz system (GLS) and hyperbolic generalized Lorenz system (HGLS) along with their canonical forms (GLCF, HGLCF), mostly presented earlier, by deriving explicit state transformation formulas to prove the equivalence between GLS and GLCF, as well as between HGLS and HGLCF. Consequently, complete formulations of the generalized Lorenz canonical systems and forms, and their hyperbolic settings, are obtained and presented. Only potentially chaotic systems are classified, which significantly helps clarify the respective canonical forms. To do so, some tools for systems to exclude chaotic behavior are developed, which are interesting in their own right for general dynamical systems theory. The new insight may inspire future investigations of generalized and canonical formulations of some other types of chaotic systems.

Keywords: Generalized Lorenz system; generalized Lorenz canonical form; hyperbolic generalized Lorenz system; hyperbolic generalized Lorenz canonical form.

1. Introduction

In this paper, the Generalized Lorenz Canonical Form (GLCF) of the Generalized Lorenz system (GLS) is revisited, which was introduced first in [Čelikovský & Vaněček, 1994] and later in a more complete fashion in [Čelikovský & Chen, 2002a]. GLS includes many previously and recently studied chaotic systems, as noted in [Čelikovský & Chen, 2005]. In particular, GLS includes the Chen system [Chen & Ueta, 1999], Lü system [Lü & Chen, 2002] and, of course, the celebrated Lorenz system itself [Lorenz, 1963], which will be referred to as the *classical Lorenz system* hereafter.

The discussions below will focus on *nontrivial* systems, which will be specified later. Roughly speaking, trivial systems have only trajectories converging to equilibrium, or have unbounded trajectories, which thereby exclude interesting dynamical phenomena like bifurcation and chaos.

The motivation of this paper is to clarify various issues noted during the last two decades since the first introduction of GLCF in [Čelikovský & Chen, 2002b], where the derivation of GLCF was completed for some key parameters in only one side of their value ranges, relying on some intuitive geometric arguments. Later on, it was realized that

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some options excluded therein to characterize even nicer single GLCF were not sorted out appropriately. The only algebraic formula relating GLCF to GLS, obtained in [Čelikovský & Chen, 2005], was also one-sided with respect to a key parameter, but along the opposite direction. Notably, it was correctly argued in [Yang *et al.*, 2006] that, as such, a complete GLCF family is transformed back into only a proper subfamily of GLS. These issues will become clearer as the discussion further develops below.

The present paper removes all the imperfections and doubts that might exist and provides rigorous and explicit formulas for linear transformations and their inverses that take GLS into GLCF, and vice versa. It also clearly relates parameters of GLS with parameters of GLCF, expressing the latter explicitly by the former. This was not quite completed in [Čelikovský & Chen, 2002a], although explicit expressions of GLS parameters were already given in [Čelikovský & Chen, 2005] using those of GLCF. It should be noted, however, that those expressions were not invertible and therefore cannot be used as an alternative for the novel treatment of the present paper.

In addition, this paper provides a canonical form for some conjugate systems, called the Hyperbolic Generalized Lorenz System (HGLS) in [Čelikovský & Chen, 2002b] and [Čelikovský & Chen, 2005]. This system differs from GLS only by a single minus sign in its quadratic part, which is replaced by a plus sign. Originally, this case was included into GLCF in [Čelikovský & Chen, 2005] as it covers some missing parameter cases. Nevertheless, as will become clearer below, essentially HGLS has a different structure and some different properties as compared to GLS.

In the present paper, it will be shown that there are some nontrivial cases of HGLS that are not covered by the generic forms derived in [Čelikovský & Chen, 2005]. For that reason, this paper presents two separate settings for GLS and HGLS, respectively, besides the above-mentioned GLCF, and a new but analogous Hyperbolic Generalized Lorenz Canonical Form (HGLCF). Consequently, a nice duality-like analogy between the two cases will be discussed in detail. To prove the triviality of some cases in both GLS and HGLS, the original technique is further developed herein, which will also be of interest in studying the general dynamical system theory. Briefly, triviality is inferred by finding a smooth function (not necessarily a sign-definite one) having negative definite time derivative along the system trajectories. Here, one actually may not conclude any particular property of the respective system trajectories; instead, it shows that any bounded trajectory must converge to a fixed point thereby implying the triviality.

Last but not least, it is noted that this paper aims to provide precise classifications of a large family of systems, which have been extensively studied especially regarding their chaotic attractors with a lot of analysis and simulations. Therefore, this paper will only contain classifications and their mathematical derivations, but not to include tedious simulations of chaos, bifurcations, oscillations, stability, and so forth, which can be easily found from the existing literature.

The rest of the paper is organized as follows. The next section reviews some concepts and definitions and provides some preliminary results, which are needed later on and are also of interest on its own right for studying general dynamical systems theory. Section 3 presents the main results about GLCF and HGLCF and their equivalence to GLS and HGLS, respectively. It also provides bibliographical information with detailed comparisons of complete results. In particular, Sec. 3.4 studies in subtle detail the special but important case connecting GLCF and HGLCF. Conclusions and research outlooks are presented in the final section.

2. Definitions and Preliminary Results

First, recall some concepts and definitions.

Definition 2.1. The generalized Lorenz system (GLS) is a three-dimensional dynamical system with real parameters $a_{11}, a_{12}, a_{21}, a_{22}, \lambda_3$ given in the following form of ordinary differential equations (ODEs):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + x_1\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \tag{1}$$

where

$$a_{11}a_{22} - a_{12}a_{21} < 0, \quad a_{11} + a_{22} < 0, \quad \lambda_3 < 0. \tag{2}$$

Remark 2.1. GLS [(1) and (2)] contains, but more importantly, generalizes the classical Lorenz system, in the sense that it not only includes the classical Lorenz system with $a_{11} = -\sigma$, $a_{12} = \sigma$, $a_{21} = r$, $a_{22} = -1$, $\lambda_3 = -b$, but also represents all the main structural features of the classical Lorenz system. Specifically, system (1) consists of a linear part and a quadratic part, where the latter is exactly the same as that of the classical Lorenz system, while the former has the same block triangular structure as the classical Lorenz system but the conditions in (2) guarantee that the eigenvalues of the two-dimensional block-matrix have one being positive and another being negative. It can be easily observed that violating any inequality in (2) would exclude some qualitative "signatures" of the classical Lorenz system. In fact, system (1) is the largest possible form that can be considered as a *generalized* Lorenz system in the above sense of structural features.

As noted in the Introduction, the following complementary system was already introduced in [Čelikovský & Chen, 2002b] and [Čelikovský & Chen, 2005].

Definition 2.2. The Hyperbolic Generalized Lorenz System (HGLS) is a three-dimensional dynamical system with real parameters a_{11}, a_{12}, a_{21}, a_{22}, λ_3, described by the following system of ODEs:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + x_1\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \tag{3}$$

where

$$a_{11}a_{22} - a_{12}a_{21} < 0, \quad a_{11} + a_{22} < 0, \quad \lambda_3 < 0. \tag{4}$$

HGLS [(3) and (4)] looks very similar to [GLS (1) and (2)], with the only sign difference in a number 1. However, this difference has significant consequences. First, some of their attractors were shown in [Čelikovský & Chen, 2002b] and [Čelikovský, 2004], which have quite different appearances. Unlike GLS, the quadratic part here no longer imposes rotational dynamics, since the matrix of the quadratic part has a pair of eigenvalues ± 1 and, from this perspective, it is named "hyperbolic".

Definition 2.3. A dynamical system consisting of smooth ODEs is said to be *trivial* if any of its trajectories either is unbounded or converges to an equilibrium point of the system; otherwise, it is *nontrivial*.

Remark 2.2. A trivial dynamical system is never chaotic, or quasi-periodic, or periodic. As trivial systems are not interesting for studying complex dynamics, they are excluded from the present study.

The following results are useful to show the triviality of a system. It is one of the main contributions of the present paper, which may also be useful for excluding chaotic behavior of a given system if it is not of interest.

Theorem 1. *Consider a smooth dynamical system, $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, and assume that there exists a smooth function $V(x)$ satisfying*

$$\dot{V}(x) \le 0, \quad \forall\, x \in \mathbb{R}^n, \quad \dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x).$$

Moreover, assume that the largest forward invariant subset of the set

$$\{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$$

is a single point. Then, this system is trivial.

Proof. This can be verified by mimicking the proof of the well-known LaSalle principle via contradiction. Indeed, assume that the system is nontrivial. Then, by the definition of the nontriviality, there exists a bounded trajectory $x(t)$, which does not converge to any single point. Consider the time function $V(x(t))$, where $V(x)$ is defined in the theorem statements. By assumption, $V(x(t))$ exists, is smooth, and is bounded and nonincreasing, therefore converges to a finite constant. Further, $V(x(t))$ is equicontinuous, since $x(t)$ is bounded, so is $\dot{V}(x(t))$ due to the smoothness of both $f(x)$ and $V(x)$. As a consequence, the real scalar function $\dot{V}(x(t))$ converges to zero.[1] Therefore, by the continuity argument, $x(t)$ converges to the set $\{x \in \mathbb{R}^n \,|\, \dot{V}(x) = 0\}$. According to the basic theory of dynamical systems it also converges to an invariant set therein, since its ω-limit set is invariant. Thus, by the theorem assumptions, $x(t)$ converges to a single point, which is a contradiction. This completes the proof. ■

The following result can be similarly proved, which provides a useful tool for excluding chaotic behaviors of a given dynamical system.

Corollary 2.1. *Consider the smooth dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, and assume that there exists a smooth function $V(x)$ satisfying*

$$\dot{V}(x) \leq 0, \quad \forall\, x \in \mathbb{R}^n, \quad \dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x).$$

Assume also that the largest forward invariant subset of the set

$$\{x \in \mathbb{R}^n \,|\, \dot{V}(x) = 0\}$$

belongs to a two-dimensional smooth embedded submanifold of $\mathbb{R}^n$. Then, any attractor of the above system belongs to this smooth manifold and, therefore, the system cannot generate chaotic behavior by virtue of the well-known Poincaré–Bendixson theorem.

Remark 2.3. It is important to emphasize that, in both of the above results, the function $V(x)$ can be *any* smooth function, even *sign indefinite*. As a matter of fact, typically sign-indefinite functions are used to prove the triviality of a system. The celebrated LaSalle invariance principle proves the asymptotic stability of a system under the same assumptions as in Theorem 1, with also assumption that the system may only have bounded behavior. Nevertheless, to show the latter one usually requires another positive definite Lyapunov function having negative semi-definite derivative along the system trajectories. As a consequence, sign-indefinite functions are rarely applied, when the LaSalle invariance principle is used to prove the asymptotic stability. In such a way, although technically Theorem 1 uses the same arguments as the proof of the LaSalle invariance principle, it presents a good possibility to prove the triviality of a system. Even when it is uncertain whether a particular system trajectory is bounded or not, Theorem 1 simply shows that the only option for the trajectory to be bounded is its convergence to an equilibrium.

Lemma 1. *GLS [(1) and (2)] is trivial if $a_{12} = 0$, or $a_{21} = 0$, $a_{11} > 0$.*

Proof

Case 1. $a_{12} = 0$. This case is obvious, since the first equation becomes $\dot{x}_1 = a_{11}x_1$ and $a_{11} \neq 0$ due to the first inequality in (2). If $a_{11} < 0$, then $x_1(t)$ tends to zero and the remaining components behave trivially as a linear system. If $a_{11} > 0$ and $x_1(0) \neq 0$, then all such system trajectories diverge to infinity. If $x_1(0) = 0$, then $x(t) \equiv 0$ and the remaining components behave as a linear system.

Case 2. $a_{21} = 0$, $a_{11} > 0$. This case can be proved using Theorem 1 with the sign-indefinite function

$$V(x) = \frac{(-x_1^2 + Rx_2^2 + Rx_3^2)}{2}, \quad R > -\frac{a_{12}^2}{4a_{11}a_{22}}.$$

Indeed, one has

$$\begin{aligned}\dot{V} &= -a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}Rx_2^2 + \lambda_3 Rx_3^2 \\ &\quad - Rx_2x_1x_3 + Rx_3x_1x_2 \\ &= -a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}Rx_2^2 + \lambda_3 Rx_3^2.\end{aligned}$$

Note that $a_{11} > 0$ by the lemma assumption. So, from $a_{11} + a_{22} < 0$ of (2), one has $a_{22} < 0$. Also, $\lambda_3 < 0$ in (2). Consequently,

$$\dot{V} = [x_1, x_2]\begin{bmatrix} -a_{11} & \dfrac{a_{12}}{2} \\ \dfrac{a_{12}}{2} & Ra_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \lambda_3 Rx_3^2,$$

$$\det\begin{bmatrix} -a_{11} & \dfrac{a_{12}}{2} \\ \dfrac{a_{12}}{2} & Ra_{22} \end{bmatrix} = -Ra_{11}a_{22} - \frac{a_{12}^2}{4} > 0,$$

[1]This is sometimes called Barbalat's lemma in the control systems' literature. But in mathematics it was known earlier without a specific name, which is merely the following property: if the integral of an equicontinuous function $\phi(t)$, defined on $[t_0, \infty)$, $t_0 \in \mathbb{R}$, converges to a constant as $t \to \infty$, then $\lim_{t\to\infty} \phi(t) = 0$. Notably, this fact was mentioned in LaSalle's seminal paper [LaSalle, 1968], stating for the first time the important result that was called later by others as the LaSalle invariance principle.

for the selection of $R > -a_{12}^2/(4a_{11}a_{22})$. Now, since $-a_{11} < 0$, it follows from the well-known criterion for negative definiteness of a symmetric matrix, and $\lambda_3 < 0$, that $\dot{V}(x) < 0$ for all $x \neq 0$. To this end, applying Theorem 1 leads to the conclusion that GLS (1) with $a_{21} = 0$, $a_{11} > 0$, is trivial. ■

Lemma 2. *HGLS [(3) and (4)] is trivial if* $a_{12} = 0$, *or* $a_{21} = 0$, $a_{22} > 0$.

Proof

Case 1. $a_{12} = 0$. This case is obvious, since the first equation becomes $\dot{x}_1 = a_{11}x_1$ with $a_{11} \neq 0$ due to the first inequality of (4) in Definition 2.2. If $a_{11} < 0$, then $x_1(t)$ tends to zero while the remaining components behave as a trivial linear system. If $a_{11} > 0$ and $x_1(0) \neq 0$, then all system trajectories diverge to infinity. If $x_1(0) = 0$, then $x(t) \equiv 0$ and the other components behave as a trivial linear system.

Case 2. $a_{21} = 0$, $a_{22} > 0$. This case can be proved using Theorem 1 with the sign-indefinite function

$$V(x) = \frac{x_1^2 - Rx_2^2 + Rx_3^2}{2}, \quad R > -\frac{a_{12}^2}{(4a_{11}a_{22})}.$$

Indeed, it can be easily verified that

$$\begin{aligned}\dot{V} &= a_{11}x_1^2 + a_{12}x_1x_2 - a_{22}Rx_2^2 + \lambda_3 Rx_3^2 \\ &\quad - Rx_2x_1x_3 + Rx_3x_1x_2 \\ &= a_{11}x_1^2 + a_{12}x_1x_2 - a_{22}Rx_2^2 + \lambda_3 Rx_3^2.\end{aligned}$$

Note that $a_{11} < 0$ by the lemma assumption of $a_{22} > 0$ and because $a_{11} + a_{22} < 0$ by the second inequality of (4) in Definition 2.2. Also, $\lambda_3 < 0$. Moreover, it follows from the above computations that

$$\dot{V} = [x_1, x_2]\begin{bmatrix} a_{11} & \dfrac{a_{12}}{2} \\ \dfrac{a_{12}}{2} & -Ra_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} + \lambda_3 Rx_3^2,$$

$$\det\begin{bmatrix} a_{11} & \dfrac{a_{12}}{2} \\ \dfrac{a_{12}}{2} & -Ra_{22}\end{bmatrix} = -Ra_{11}a_{22} - \frac{a_{12}^2}{4} > 0,$$

for the selection of $R > -a_{12}^2/(4a_{11}a_{22})$. Since $a_{11} < 0$ by the lemma assumption of $a_{22} > 0$ and $a_{11} + a_{22} < 0$ by the second inequality of (4) in Definition 2.2, using the well-known criterion for negative definiteness of a symmetric matrix, and $\lambda_3 < 0$, one has $\dot{V}(x) < 0$ for all $x \neq 0$. To this end, applying Theorem 1 leads to the conclusion that HGLS [(3) and (4)] with $a_{21} = 0$, $a_{22} > 0$, is trivial. ■

To conclude this section, for later convenience, some notation and properties are summarized from both GLS [(1) and (2)] and HGLS [(3) and (4)], as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}, \quad \det\begin{bmatrix} a_{11} - \lambda_i & a_{12} \\ a_{21} & a_{22} - \lambda_i\end{bmatrix} = 0,$$
$$i = 1, 2, \quad \lambda_1 > 0, \quad \lambda_2 < 0, \tag{5}$$

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\{a_{11} + a_{22} \\ &\quad + \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}\} \\ &= \frac{1}{2}\{a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\} \\ &> 0,\end{aligned} \tag{6}$$

$$\begin{aligned}\lambda_2 &= \frac{1}{2}\{a_{11} + a_{22} \\ &\quad - \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}\} \\ &= \frac{1}{2}\{a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\} \\ &< 0.\end{aligned} \tag{7}$$

Further, introduce the following key parameter $\tau \neq -1$, defined for both GLS and HGLS, when $a_{11} \neq \lambda_1$:

$$\begin{aligned}\tau &:= -\frac{\lambda_2 - a_{11}}{\lambda_1 - a_{11}} \\ &= \frac{4a_{12}a_{21}}{[a_{11} - a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}]^2}, \\ &\qquad \tau \neq -1, \ \ a_{11} \neq \lambda_1.\end{aligned} \tag{8}$$

Note that $\tau = -1$ implies $\lambda_1 = \lambda_2$, contradicting both definitions of GLS [(1) and (2)] and of HGLS [(3) and (4)]. Indeed, by the assumptions (2) and (4), one has $\lambda_1 > 0$, $\lambda_2 < 0$. Otherwise, every value of $\tau \neq -1$ can be obtained for some values of parameters $a_{11}, a_{12}, a_{21}, a_{22}$ in (8).

3. Main Results

The canonical forms for both GLS and HGLS are established here. Only nontrivial systems will be considered and therefore $a_{12} \neq 0$ will be assumed in the sequel by virtue of Lemmas 1 and 2.

Note that, for GLS, the case of $a_{12} \neq 0$, $a_{21} = 0$, $a_{11} > 0$ is trivial by Lemma 1. Thus, by formula (6), possible nontriviality implies that $a_{11} \neq \lambda_1$. Indeed, $a_{11} = \lambda_1$ implies by (6) both $a_{12}a_{21} = 0$ and $a_{11} = \lambda_1 > 0$. Therefore, the key parameter τ is always well defined by (8) for nontrivial GLS [(1) and (2)]. In particular, there exists a nontrivial case with $\tau = 0$ when $a_{21} = 0$ and $a_{11} = \lambda_2$, which leads to the Lü system [Lü & Chen, 2002]. Moreover, $\mathrm{sign}(\tau) = \mathrm{sign}(a_{12}a_{21})$, and the GLS with $\tau > 0$ is the classical Lorenz system [Lorenz, 1963], while the GLS with $\tau < 0$ is the Chen system [Chen & Ueta, 1999].

Finally, for HGLS, the case of $a_{12} \neq 0$, $a_{21} = 0$, $a_{22} > 0$ is trivial by Lemma 2 as well, with $\tau = 0$ for this case. Yet, the case of $a_{12} \neq 0$, $a_{21} = 0$, $a_{11} > 0$ may not be trivial. This case corresponds to the *hyperbolic Lü system* to be discussed further later. Since the key parameter τ is not defined for the hyperbolic Lü system by (8), this case will be handled separately when deriving respective canonical forms. Again, for HGLS, one has $\mathrm{sign}(\tau) = \mathrm{sign}(a_{12}a_{21})$ and it is natural to call the case with $\tau > 0$ the *hyperbolic Lorenz system*, while the case with $\tau < 0$, the *hyperbolic Chen system*.

3.1. *Transformations of (hyperbolic) generalized Lorenz systems*

This subsection presents preparatory results giving unified state transformations of possibly nontrivial GLS and HGLS. Their nontriviality requires, besides $a_{12} \neq 0$, one of the following options:

$$a_{11} \neq \lambda_1 \Leftrightarrow a_{21} \neq 0 \vee (a_{21} = 0 \wedge a_{11} < 0), \tag{9}$$

$$a_{11} = \lambda_1 \Leftrightarrow a_{21} = 0 \wedge a_{11} > 0. \tag{10}$$

First, consider case (9), for which the following result is established.

Theorem 2. *Consider both GLS [(1) and (2)] and HGLS [(3) and (4)] with $a_{12} \neq 0$ and assume (9) holds. Define a linear change of coordinates from the original coordinates $x \in \mathbb{R}^n$ to the new coordinates $z \in \mathbb{R}^3$ as follows*[2]:

$$x = Tz, \quad T = |\tau+1|^{1/2} \begin{bmatrix} 1 & -1 & 0 \\ \dfrac{\lambda_1 - a_{11}}{a_{12}} & -\dfrac{\lambda_2 - a_{11}}{a_{12}} & 0 \\ 0 & 0 & \dfrac{\lambda_1 - a_{11}}{a_{12}}|\tau+1|^{1/2} \end{bmatrix}, \tag{11}$$

$$z = T^{-1}x, \quad T^{-1} = (\tau+1)^{-1}|\tau+1|^{-1/2} \begin{bmatrix} \tau & \dfrac{a_{12}}{\lambda_1 - a_{11}} & 0 \\ -1 & \dfrac{a_{12}}{\lambda_1 - a_{11}} & 0 \\ 0 & 0 & \dfrac{a_{12}}{\lambda_1 - a_{11}}|\tau+1|^{-1/2}(\tau+1) \end{bmatrix}. \tag{12}$$

Then, [(11) and (12)] transform GLS [(1) and (2)] and HGLS [(3) and (4)] together into the following form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + (z_1 - z_2) \begin{bmatrix} 0 & 0 & \mu\,\mathrm{sign}(\tau+1) \\ 0 & 0 & \mu\,\mathrm{sign}(\tau+1) \\ 1 & \tau & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad \tau \neq -1, \tag{13}$$

where τ, λ_1, λ_2 are given by (8), (6), (7), respectively, $\mu = -1$ for GLS and $\mu = 1$ for HGLS.

[2]Here, τ, λ_1, λ_2 are to be substituted from (8), (6), (7), respectively. In such a way, transformations (11) and (12) are actually expressed via GLS (1) and HGLS (3) parameters while keeping the formulas in (11) and (12) reasonably short and compact.

Proof. First, for convenience later, consider a unified form of both GLS and HGLS:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+x_1\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \mu\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},$$

$$a_{12}\neq 0,\ \ a_{11}\neq\lambda_1, \tag{14}$$

which is GLS if $\mu=-1$ and is HGLS if $\mu=1$. Recall that $a_{11}\neq\lambda_1$ as assumed by (9).

Since $a_{11}\neq\lambda_1$, using $x_1=|\tau+1|^{1/2}(z_1-z_2)$, obtained from (11), and by (14), (11), (12), one has

$$\begin{aligned}\dot{z}&=T^{-1}\dot{x}\\ &=T^{-1}\begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}Tz\\ &\quad+|\tau+1|^{1/2}(z_1-z_2)T^{-1}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \mu\\ 0 & 1 & 0\end{bmatrix}Tz,\end{aligned} \tag{15}$$

where $|\tau+1|^{1/2}(z_1-z_2)$ is a scalar, therefore can be moved to the front of the second additive term.

To evaluate the first additive term in (15), specify the transform T as follows:

$$\begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}T=|\tau+1|^{1/2}\begin{bmatrix}a_{11}+\lambda_1-a_{11} & -a_{11}-\lambda_2+a_{11} & 0\\ a_{21}+a_{22}\dfrac{\lambda_1-a_{11}}{a_{12}} & -a_{21}-a_{22}\dfrac{\lambda_2-a_{11}}{a_{12}} & 0\\ 0 & 0 & \lambda_3\dfrac{\lambda_1-a_{11}}{a_{12}}|\tau+1|^{1/2}\end{bmatrix},$$

$$\begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}T=|\tau+1|^{1/2}\begin{bmatrix}\lambda_1 & -\lambda_2 & 0\\ \dfrac{\lambda_1 a_{22}+a_{21}a_{12}-a_{11}a_{22}}{a_{12}} & \dfrac{-\lambda_2 a_{22}-a_{21}a_{12}+a_{11}a_{22}}{a_{12}} & 0\\ 0 & 0 & \lambda_3\dfrac{\lambda_1-a_{11}}{a_{12}}|\tau+1|^{1/2}\end{bmatrix}.$$

Since the sum of all eigenvalues is equal to the trace of the matrix, and their product to the determinant, one has $a_{11}a_{22}-a_{21}a_{12}=\lambda_1\lambda_2$, $a_{22}-\lambda_2=\lambda_1-a_{11}$ and $a_{22}-\lambda_1=\lambda_2-a_{11}$, which together yield

$$\begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}T=|\tau+1|^{1/2}\begin{bmatrix}\lambda_1 & -\lambda_2 & 0\\ \lambda_1\dfrac{-\lambda_2+a_{22}}{a_{12}} & \lambda_2\dfrac{\lambda_1-a_{22}}{a_{12}} & 0\\ 0 & 0 & \lambda_3\dfrac{\lambda_1-a_{11}}{a_{12}}|\tau+1|^{1/2}\end{bmatrix},$$

$$\begin{bmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda_3\end{bmatrix}T=|\tau+1|^{1/2}\begin{bmatrix}\lambda_1 & -\lambda_2 & 0\\ \lambda_1\dfrac{\lambda_1-a_{11}}{a_{12}} & \lambda_2\dfrac{a_{11}-\lambda_2}{a_{12}} & 0\\ 0 & 0 & \lambda_3\dfrac{\lambda_1-a_{11}}{a_{12}}|\tau+1|^{1/2}\end{bmatrix}=T\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}.$$

Summarizing the above, by the last several equalities for the first group of terms in (15), one arrives at

$$T^{-1}\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{16}$$

Next, to evaluate the second group of terms in (15), by (11) and (12), one has

$$T^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & 1 & 0 \end{bmatrix} T = (\tau+1)^{-1}\begin{bmatrix} \tau & \dfrac{a_{12}}{\lambda_1 - a_{11}} & 0 \\ -1 & \dfrac{a_{12}}{\lambda_1 - a_{11}} & 0 \\ 0 & 0 & \dfrac{a_{12}(\tau+1)}{\lambda_1 - a_{11}}|\tau+1|^{-1/2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -1 & 0 \\ \dfrac{\lambda_1 - a_{11}}{a_{12}} & -\dfrac{\lambda_2 - a_{11}}{a_{12}} & 0 \\ 0 & 0 & \dfrac{\lambda_1 - a_{11}}{a_{12}}|\tau+1|^{1/2} \end{bmatrix}$$

$$= (\tau+1)^{-1}\begin{bmatrix} \tau & \dfrac{a_{12}}{\lambda_1 - a_{11}} & 0 \\ -1 & \dfrac{a_{12}}{\lambda_1 - a_{11}} & 0 \\ 0 & 0 & \dfrac{a_{12}(\tau+1)}{\lambda_1 - a_{11}}|\tau+1|^{-1/2} \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu\dfrac{\lambda_1 - a_{11}}{a_{12}}|\tau+1|^{1/2} \\ \dfrac{\lambda_1 - a_{11}}{a_{12}} & -\dfrac{\lambda_2 - a_{11}}{a_{12}} & 0 \end{bmatrix}$$

$$= (\tau+1)^{-1}\begin{bmatrix} 0 & 0 & \mu|\tau+1|^{1/2} \\ 0 & 0 & \mu|\tau+1|^{1/2} \\ |\tau+1|^{-1/2}(\tau+1) & -\dfrac{\lambda_2 - a_{11}}{\lambda_1 - a_{11}}|\tau+1|^{-1/2}(\tau+1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \mu|\tau+1|^{-1/2}\,\mathrm{sign}(\tau+1) \\ 0 & 0 & \mu|\tau+1|^{-1/2}\,\mathrm{sign}(\tau+1) \\ |\tau+1|^{-1/2} & -\dfrac{\lambda_2-a_{11}}{\lambda_1-a_{11}}|\tau+1|^{-1/2} & 0 \end{bmatrix}$$

$$= |\tau+1|^{-1/2}\begin{bmatrix} 0 & 0 & \mu\,\mathrm{sign}(\tau+1) \\ 0 & 0 & \mu\,\mathrm{sign}(\tau+1) \\ 1 & \tau & 0 \end{bmatrix}.$$

The last two equalities are obtained by using

$$\begin{aligned}(\tau+1)^{-1}|\tau+1|^{1/2} &= (\tau+1)^{-1}|\tau+1||\tau+1|^{-1/2} \\ &= \mathrm{sign}(\tau+1)|\tau+1|^{-1/2}\end{aligned}$$

and using the definition of τ in (8), namely

$$\tau = -\frac{\lambda_2-a_{11}}{\lambda_1-a_{11}}, \quad \tau \neq -1.$$

Summarizing the above, along with the previous several equalities for the second additive group of terms in (15), one finally obtains

$$T^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & 1 & 0 \end{bmatrix} T = |\tau+1|^{-1/2}\begin{bmatrix} 0 & 0 & \mu\,\mathrm{sign}(\tau+1) \\ 0 & 0 & \mu\,\mathrm{sign}(\tau+1) \\ 1 & \tau & 0 \end{bmatrix}. \quad (17)$$

The proof is then completed by substituting (16) and (17) into (15), which gives (13). ■

Second, consider the case of (10). One has the following result.

Theorem 3. *Let $a_{12} \neq 0$ and assume (10) holds, i.e. $a_{21} = 0$, $a_{11} > 0$, $a_{22} < 0$. Consider the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + x_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad (18)$$

Then, (18) is state-equivalent to the following system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + (z_1-z_2)\begin{bmatrix} 0 & 0 & \mu \\ 0 & 0 & \mu \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad \lambda_1 \neq \lambda_2, \quad (19)$$

where $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, and the respective state-transformations are

$$x_1 = z_1 - z_2, \quad x_2 = \frac{\lambda_1-\lambda_2}{a_{12}}z_2, \quad x_3 = \frac{\lambda_1-\lambda_2}{a_{12}}z_3, \quad (20)$$

$$z_1 = x_1 + \frac{a_{12}}{\lambda_1-\lambda_2}x_2, \quad z_2 = \frac{a_{12}}{\lambda_1-\lambda_2}x_2, \quad z_3 = \frac{a_{12}}{\lambda_1-\lambda_2}x_3. \quad (21)$$

Proof. Recall the theorem assumption of ($\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$). It follows from (18), (20) and (21) that

$$\begin{aligned}\dot{z}_1 &= \dot{x}_1 + \frac{a_{12}}{\lambda_1-\lambda_2}\dot{x}_2 \\ &= \lambda_1 x_1 + a_{12}x_2 + \frac{a_{12}}{\lambda_1-\lambda_2}(\lambda_2 x_2 + \mu x_1 x_3) \\ &= \lambda_1(z_1-z_2) + a_{12}\frac{\lambda_1-\lambda_2}{a_{12}}z_2 + \frac{a_{12}}{\lambda_1-\lambda_2} \\ &\quad \times \left(\lambda_2\frac{\lambda_1-\lambda_2}{a_{12}}z_2 + \mu(z_1-z_2)\frac{\lambda_1-\lambda_2}{a_{12}}z_3\right) \\ &= \lambda_1 z_1 + \mu(z_1-z_2)z_3,\end{aligned}$$

$$\begin{aligned}\dot z_2 &= \frac{a_{12}}{\lambda_1-\lambda_2}\dot x_2\\ &= \frac{a_{12}}{\lambda_1-\lambda_2}\left(\lambda_2\frac{\lambda_1-\lambda_2}{a_{12}}z_2+\mu(z_1-z_2)\frac{\lambda_1-\lambda_2}{a_{12}}z_3\right)\\ &= \lambda_2 z_2+\mu(z_1-z_2)z_3,\\ \dot z_3 &= \frac{a_{12}}{\lambda_1-\lambda_2}\dot x_3\\ &= \frac{a_{12}}{\lambda_1-\lambda_2}\left(\lambda_3\frac{\lambda_1-\lambda_2}{a_{12}}z_3+(z_1-z_2)\frac{\lambda_1-\lambda_2}{a_{12}}z_2\right)\\ &= \lambda_3 z_3+(z_1-z_2)z_2,\end{aligned}$$

which gives (19). The proof is thus completed. ■

Lemma 3. *System (13) with $\tau < -1$ and $\mu = -1$, or with $\tau \in (-1, 0]$ and $\mu = 1$, is trivial.*

Proof. First, for $\tau < -1$ and $\mu = -1$, one has $\mu\,\mathrm{sign}(\tau+1) = 1$, and the same equality holds for $\tau \in (-1, 0]$ and $\mu = 1$. So, under the lemma assumptions, the system (13) becomes

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + (z_1-z_2)\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & \tau & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}.$$

Next, consider the following sign-indefinite function:

$$V(z_1, z_2, z_3) = \frac{(-z_1^2-\tau z_2^2+z_3^2)}{2}.$$

From the above system with $V(z_1, z_2, z_3)$, one obtains

$$\begin{aligned}\dot V &= -\lambda_1 z_1^2-\lambda_2\tau z_2^2+\lambda_3 z_3^2\\ &\quad+(z_1-z_2)(-z_1z_3-\tau z_2 z_3+z_1 z_3+\tau z_2 z_3)\\ &= -\lambda_1 z_1^2-\lambda_2\tau z_2^2+\lambda_3 z_3^2.\end{aligned}$$

If $\tau \neq 0$ then $\dot V < 0$ for all $[z_1, z_2, z_3]^\top \neq 0$, since $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, $\tau < 0$, and therefore $\lambda_2\tau > 0$. By Theorem 1, the system (13) with $\tau < -1$ and $\mu = -1$, or with $\tau \in (-1, 0)$ and $\mu = 1$, is trivial, as claimed.

The case of $\tau = 0$ and $\mu = 1$ can be verified by similar arguments as the proof of Theorem 1, since the set $\dot V$ here is given by $z_1 = z_3 = 0$ and the dynamics on this set are described by $\dot z_1 = 0$, $\dot z_2 = \lambda_2 z_2$, $\dot z_3 = 0$, where $\lambda_2 < 0$. So, any bounded trajectory will converge to the set where the restricted invariant dynamics converge to the origin; therefore, the bounded trajectory converges to the origin as well. ■

3.2. *Generalized Lorenz canonical form*

The canonical form of any nontrivial [GLS (1) and (2)] is specified by the following theorem.

Theorem 4. *Every nontrivial GLS [(1) and (2)] is state-equivalent, under linear transformation (11) and (12), to the following Generalized Lorenz Canonical Form (GLCF):*

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + (z_1-z_2)\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & -1\\ 1 & \tau & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}, \quad \tau > -1, \tag{22}$$

where τ is given by (8), and λ_1, λ_2 are given by (6), (7).

Proof. Obviously, $a_{12} \neq 0$ and (9) should hold; otherwise, GLS is trivial by Lemma 1. Thus, assumptions of Theorem 2 are satisfied and GLS is state-equivalent to (13) with $\mu = -1$. Further, note that a nonsingular linear change of coordinates of the system will not affect its triviality and, therefore, all trivial cases of (13) should be excluded. By Lemma 3, the cases of (13) with $\mu = -1$, $\tau < -1$ are trivial, while (13) with $\mu = -1$, $\tau > -1$ gives (22). ■

Remark 3.1. Theorem 4 was partly formulated and proved in [Čelikovský & Chen, 2002a]. The proof provided there was rather complicated using some intuitive geometric arguments without explicitly showing algebraic state transformations (11) and (12), which are a new contribution of the present paper, showing the state-equivalence in a pure algebraic fashion and rigorously. Another novelty of the present paper is the closed-form formulas for the relations between the parameters of GLS [(1) and

(2)] and that of its canonical form (22); specifically, τ is given by (8), and λ_1, λ_2 by (6), (7), with $\lambda_3 < 0$. To be convenient in referring to these in the sequel, all the formulas are rewritten in a more compact form as follows:

$$\begin{aligned}\tau &= -(\lambda_2 - a_{11})(\lambda_1 - a_{11})^{-1}\\ &= 4a_{12}a_{21}(a_{11} - a_{22}\\ &\quad - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}})^{-2},\\ \lambda_1 &= \frac{1}{2}(a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}),\\ \lambda_2 &= \frac{1}{2}(a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}),\\ \lambda_3 &= \lambda_3.\end{aligned} \tag{23}$$

Recall that nontriviality requires $a_{12} \neq 0$ and, meanwhile, either $a_{21} \neq 0$ or $a_{11} < 0$. This ensures that all divisions in (23) are by nonzero quantities, as already explained in detail right after (8). Particular cases of GLS [(1) and (2)] and GLCF (22) are classified in Table 1.

Remark 3.2. The relations in (23) cannot be one-to-one, because GLS has five parameters while GLCF has only four. Nevertheless, in [Čelikovský & Chen, 2005] a "pseudo-inverse" parametric relation was derived, which takes the parameters of GLCF to the parameters of GLS, as follows:

$$\begin{aligned}a_{11} &= \lambda_1 + (\lambda_2 - \lambda_1)(\tau + 1)^{-1},\\ a_{12} &= -(\lambda_2 - \lambda_1)(\tau + 1)^{-1},\\ a_{21} &= (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_1)(\tau + 1)^{-1},\\ a_{22} &= \lambda_2 - (\lambda_2 - \lambda_1)(\tau + 1)^{-1},\\ \lambda_3 &= \lambda_3.\end{aligned} \tag{24}$$

Table 1. Relations between the generalized Lorenz system and its canonical form.

GLS [(1) and (2)]	GLCF (22)
$a_{12} = 0$: *trivial*	*None*
$a_{21} = 0 \wedge a_{11} \geq 0$: *trivial*	*None*
$a_{21} = 0 \wedge a_{11} < 0$: *Lü system*	$\tau = 0$
$a_{21}a_{12} > 0$: *Lorenz system*	$\tau > 0$
$a_{21}a_{12} < 0$: *Chen system*	$-1 < \tau < 0$

Here, "pseudo-inverse" simply means that substituting a_{11}, a_{12}, a_{21}, a_{22}, λ_3 from (24) to the right-hand side of (23) gives, by straightforward but tedious computations, the identities $\lambda_1 = \lambda_1$, $\lambda_2 = \lambda_2$, $\lambda_3 = \lambda_3$, $\tau = \tau$. Yet, substituting λ_1, λ_2, λ_3, τ from (23) to the right-hand side of (24) does not give, in general, similar identities for the parameters $a_{11}, a_{12}, a_{21}, a_{22}, \lambda_3$ of the "starting" GLS (1). Instead, it gives GLS (1) with parameters $\bar{a}_{11}$, $\bar{a}_{12}$, $\bar{a}_{21}$, $\bar{a}_{22}$, $\bar{\lambda}_3$ as follows:

$$\begin{aligned}&\bar{a}_{11} = a_{11}, \quad \bar{a}_{22} = a_{22}, \quad \bar{\lambda}_3 = \lambda_3,\\ &\bar{a}_{12} = -\frac{1}{2}(a_{11} - a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}),\\ &\bar{a}_{21} = -2(a_{11} - a_{22}\\ &\qquad - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}})^{-1}a_{12}a_{21}.\end{aligned} \tag{25}$$

Parameter relations (24), in combination with (11), (12), give the following state transformation:

$$\begin{aligned}x_1 &= |\tau + 1|^{1/2}(z_1 - z_2),\\ x_2 &= |\tau + 1|^{1/2}(z_1 + \tau z_2),\\ x_3 &= |\tau + 1|z_3,\end{aligned} \tag{26}$$

$$\begin{aligned}z_1 &= (\tau + 1)^{-1}|\tau + 1|^{-1/2}(\tau x_1 + x_2),\\ z_2 &= -(\tau + 1)^{-1}|\tau + 1|^{-1/2}(x_1 - x_2),\\ z_3 &= |\tau + 1|^{-1}x_3.\end{aligned} \tag{27}$$

Note that the state transformation (26) and (27) depends only on τ, while (11) and (12) include all parameters of GLS (1).

Remark 3.3. As a side note, the above property may offer an asymmetric encryption scheme for chaos-based encryption with the starting parameters of GLS being used for a private key and the resulting parameters of GLCF being used for a public key. Indeed, a single quadruple of the latter parameters corresponds to infinitely many quintuples of the former parameters. Moreover, to reconstruct signals of the original GLS from its GLCF one needs transformation (11) and (12) requiring, in turn, knowledge of all GLS parameters. Another application of utilizing such properties is to study the generalized synchronization of distinct GLS's. For definition of the generalized synchronization, including its study in complex chaotic networks, see e.g. [Lynnyk *et al.*, 2020] and some references therein.

3.3. *Hyperbolic generalized Lorenz canonical form*

The most important new result of this paper is now presented. It shows that the canonical form of any nontrivial HGLS [(3) and (4)] is given by the unified formulation presented in the following theorem.

Theorem 5. *Every nontrivial HGLS* [(3) *and* (4)] *is state-equivalent to the following Hyperbolic Generalized Lorenz Canonical Form* (*HGLCF*):

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + (z_1 - z_2)\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & -1\\ 1 & \tau & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}, \quad \tau < -1, \tag{28}$$

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + (z_1 - z_2)\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & \tau & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}, \quad \tau > 0, \tag{29}$$

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + (z_1 - z_2)\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}, \tag{30}$$

where τ *is given by* (8) *and* λ_1, λ_2 *are given by* (6), (7). *The cases of* (28) *and* (29) *are obtained via a linear transformation* (11) *and* (12) *while the case of* (30) *via* (20) *and* (21).

Proof. The result directly follows from Theorems 2, 3 and Lemmas 2, 3, as well as the fact that a nonsingular linear change of coordinates of the system will not affect its triviality or nontriviality. Indeed, for (9), applying Theorem 2 for HGLS ($\mu = 1$) and then using Lemma 3 lead to the conclusion that only cases of (28) and (29) might be nontrivial. Furthermore, for (10), i.e. $a_{12} \neq 0$, $a_{21} = 0$, $a_{11} > 0$, $a_{22} < 0$, Theorem 3 yields (30). The remaining cases of $a_{12} = 0$ or $a_{21} = 0$, $a_{22} > 0$ are trivial by Lemma 2. ■

Remark 3.4. Note that $\lambda_1 = a_{11}$ giving the HGLCF (30) if and only if $\lambda_2 = a_{22}$, $a_{21} = 0$. This case is referred to as the *hyperbolic Lü system*, since its linear part has a similar upper triangular structure as the Lü system [Lü & Chen, 2002]. Note also that formula (8), which defines τ, is meaningful only for $\lambda_1 \neq a_{11}$. For GLS, $\lambda_1 \neq a_{11}$ was guaranteed by the nontriviality assumption based on Lemma 1. For HGLS, $\lambda_1 = a_{11}$ is no longer excluded, which changes (30) to be HGLCF of the hyperbolic Lü system.

Table 2 summarizes various cases of HGLS (1) and relates them to HGLCF. Precise relations among the parameters of possibly nontrivial cases of HGLS (3) and HGLCF are the same[3] as for GLS, namely (23).

Remark 3.5. The same precise formulas relating parameters of HGLS to respective HGLCF apply, in the same way as detailed for GLS in the previous subsection. Similarly, the pseudo-inverse transformation (24) of parameters can be applied to HGLCF as well, including respective coordinate transformations (26) and (27), depending only on τ. The only distinction is that HGLCF (30) contains the hyperbolic Lü system (18) via the specific transformations (20) and (21) respectively.

Table 2. Relations between HGLS and its canonical form HGLCF.

HGLS (3)	HGLCF
$a_{12} = 0$: *trivial*	*None*
$a_{21} = 0 \wedge a_{22} \geq 0$: *trivial*	*None*
$a_{21} = 0 \wedge a_{22} < 0$: *hyperbolic Lü system*	HGLCF (30): $\tau = \pm\infty$
$a_{21}a_{12} > 0$: *hyperbolic Lorenz system*	HGLCF (29), $\tau > 0$
$a_{21}a_{12} < 0$: *hyperbolic Chen system*	HGLCF (28), $\tau < -1$

[3]This excludes the hyperbolic Lü system, in which τ does not exist because $\lambda_1 = a_{11}$. Table 2 includes $\tau = \pm\infty$, since this case is approached as $\lambda_1 \to a_{11}$, which gives τ those two limiting "values" in (23).

Remark 3.6. Another interesting yet challenging problem remains open. To date, chaotic behaviors were not found from systems (29) and (30), and it also seems to be very difficult, if not impossible, to prove that they are trivial. Therefore, the following open problem is posted:

Either find some values of parameters $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$ and $\tau > 0$ such that systems (29) and (30) exhibit chaotic behaviors, or prove that these systems are trivial.

3.4. *The special case of $\tau = -1$*

The case of $\tau = -1$ was studied before with simulations shown in [Čelikovský, 2004]. However, this case clearly does not correspond to any of the canonical forms (22), (28); it neither corresponds to (1) and (2) nor to (3) and (4). Indeed, $\tau = -1$ only if the matrix A in (5) satisfies $\lambda_1 = \lambda_2$, which is excluded by both (2) and (4). Moreover, (13), which generates (22) and (28), does not allow $\tau = -1$, regardless of the definition for sign(0). To include $\tau = -1$ into the context of (22) and (28), one may recall the following theorem given in [Čelikovský, 2004].

Theorem 6. *The following system*

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + (z_1 - z_2) \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{31}$$

is both state- and time-scaling equivalent to the Shimizu–Morioka model [Shimizu & Morioka, 1976]

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\theta} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= x(1-z) + \lambda y, \\ \frac{\mathrm{d}z}{\mathrm{d}\theta} &= -\alpha z + x^2, \end{aligned} \tag{32}$$

through the following state transformations and time scaling:

$$\begin{aligned} x &= (\lambda_1 - \lambda_2)^{1/2}(-\lambda_1\lambda_2)^{-3/4}(z_1 - z_2), \\ y &= (\lambda_1 - \lambda_2)^{1/2}(-\lambda_1\lambda_2)^{-5/4}(\lambda_1 z_1 - \lambda_2 z_2)^{1/2}, \\ z &= (\lambda_2 - \lambda_1)(\lambda_1\lambda_2)^{-1} z_3, \\ \theta &= t(-\lambda_1\lambda_2)^{1/2}. \end{aligned} \tag{33}$$

Here, parameters of (32) and (31) are related via $\lambda = (\lambda_1 + \lambda_2)(-\lambda_1\lambda_2)^{-1/2}$, $\alpha = \lambda_3(-\lambda_1\lambda_2)^{-1/2}$.

Remark 3.7. The so-called "Liu–Liu–Liu–Liu system" [Liu *et al.*, 2004] was shown to be state- and time-scaling equivalent to the Shimizu–Morioka model (32) in Corollary 3.4 of [Čelikovský & Chen, 2005].

In view of (13) and the ambiguous specification of sign(0) therein, it is reasonable to consider yet another system possibly including also the case of $\tau = -1$, in the following form:

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + (z_1 - z_2) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{34}$$

However, this option is not so interesting, as it is a trivial system. Indeed, one has the following result.

Lemma 4. *System (34) is trivial for all cases with $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_3 > 0$.*

Proof. Theorem 1 can be directly applied, with the function $V = (-z_1^2 + z_2^2 + z_3^2)/2$, to yield

$$\begin{aligned} \dot V &= -\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 + (z_1 - z_2) \\ &\quad \times (-z_1 z_3 + z_2 z_3 + z_3 z_1 - z_3 z_2) \\ &= -\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 < 0, \end{aligned}$$

which holds for all $(z_1, z_2, z_3)^\top \neq 0$, since $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$, by the lemma assumption. ■

Remark 3.8. Notably, (13) as the unified form equivalent to both GLS ($\mu = -1$) and HGLS ($\mu = 1$) has ambiguous extensions to the case of $\tau = -1$, depending on how the *signum* function is defined at zero:

If one defines sign(0) = +1, then (13) with $\tau = -1$ becomes (31) for $\mu = -1$ and (34) for $\mu = 1$.

If one defines $\mathrm{sign}(0) = -1$, then (13) with $\tau = -1$ becomes (34) for $\mu = -1$ and (31) for $\mu = 1$.

Therefore, the situation can be seen as that both GLS-equivalent systems (with $\mu = -1$) and HGLS-equivalent systems (with $\mu = 1$) in (13) "shake hands" at $\tau = -1$. Moreover, since system (34) is trivial, it is reasonable to relate the case of $\tau = -1$ only to system (31).

In [Čelikovský & Chen, 2005], the following theorem was established.

Theorem 7. *Let $\tau \neq -1$. Then, the system*

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + (z_1 - z_2) \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{35}$$

is state-equivalent to the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{sign}(\tau + 1) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{36}$$

where the parameters of (36) are computed from the parameters of (35) by using formulas (24), and the state-equivalence transformations are given by (26) and (27).

As a matter of fact, Theorem 7 seemingly puts together Theorem 4 and the case (28) of Theorem 5. Yet, Theorem 7 uses different transformations comparing to Theorem 4 and the proof of Theorem 5.

As already noted, the cases of (29) and (30) in Theorem 5 are newly found, unknown before, which constitutes the most significant contribution of the present paper.

Table 3, presented in [Čelikovský & Chen, 2005] as "Table 1", is not as interesting as the above new result, since it does not cover all possible cases shown in Tables 1 and 2 of this paper. Nevertheless, Table 3 is included here for completeness of the presentation, which can also be viewed as a complementary interpretation for Theorems 6 and 7 of this paper.

Table 3. Relations between the generalized Lorenz system and its canonical form.

System (35)	Equivalent Systems
$\tau < -1$	HGLS (3) with $\lambda_1 < a_{11}$
$\tau = -1$	Shimizu-Morioka model
$\tau \in (-1, 0)$	GLS with $a_{12}a_{21} < 0$; Chen system
$\tau = 0$	GLS with $a_{21} = 0$; Lü system
$\tau > 0$	GLS with $a_{12}a_{21} > 0$; classical Lorenz system

It should be noted that Table 1 in [Čelikovský & Chen, 2005], namely Table 3 of this paper, was correct but was not appropriately explained, which refers only to Theorems 6 and 7 in this paper. It is important to underline that neither Table 1 in [Čelikovský & Chen, 2005] nor Table 3 here covers the cases of (29) and (30) in Theorem 5; thereby, system (35) is not simply referred to as GLCF as in [Čelikovský & Chen, 2005]. Instead, for a clearer classification, the present paper has introduced two separate canonical forms: GLCF and HGLCF.

Now, using Theorems 4 and 5, all tables are finally fully justified and explained. This is another new contribution of the present paper.

Finally, it is noted that all cases given in Table 3 allow to quite easily tune and simulate chaotic-like behaviors [Čelikovský & Chen, 2002a, 2002b; Lü & Chen, 2002; Čelikovský, 2004]. From this perspective, the above "open problem" in Remark 3.6 is indeed a challenging one, since the cases of (29) and (30) in Theorem 5 are both newly derived cases in addition to those shown in Table 3.

4. Conclusions

This paper completes the description of the (hyperbolic) generalized Lorenz systems and their canonical forms, presented earlier by Čelikovský and Chen in their joint papers cited above.

Specifically, this paper derives explicit state transformations to prove the equivalence between GLS to GLCF, which was shown in [Čelikovský & Chen, 2002a] by using geometric arguments. Moreover, it provides complete and rigorous proofs of a couple of not-so completely justified statements made in [Čelikovský & Chen, 2005]. Most significantly, it presents complete formulations of the generalized Lorenz canonical systems and forms, as well as their hyperbolic settings.

This paper also has some interesting by-products. For example, it shows how to determine the triviality or nontriviality of a dynamical system, so as to exclude or include possible chaotic behaviors, e.g. in some possibly nontrivial cases of (hyperbolic) generalized Lorenz systems. It moreover shows precise relations to the previously suggested limiting parameter transformations, with the possibility for designing asymmetric encryption schemes for chaos-based encryption. Another possible application is to use the parameter redundancy and "pseudo-inverse" parameter relation given by (24) to develop generalized synchronization in chaotic networks [Lynnyk *et al.*, 2020].

It is expected that this paper offers some insightful ideas and methodologies that could inspire future investigations of generalized and canonical formulations of some other types of nonlinear dynamical systems, especially chaotic systems.

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